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Mikhail Popov,
Beata Randrianantoanina

NARROW OPERATORS ON FUNCTION SPACES AND VECTOR LATTICES

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Mikhail Popov
Beata Randrianantoanina

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To Mila, Anita and Yola

Preface

Most classes of operators that are not isomorphic embeddings are characterized by some kind of a "smallness" condition. Narrow operators are those operators defined on function spaces that are "small" at signs, i.e. at $\{-1, 0, 1\}$-valued functions. The idea to consider such operators has led to many interesting problems that can be applied to geometric functional analysis, operator theory and vector lattices.

Narrow operators were formally defined and named by Plichko and Popov in 1990 (see [110] and [115]) for operators acting from a rearrangement invariant function F-space with an absolutely continuous norm to an F-space. However, several authors studied this type of operators earlier, including Bourgain [19] (1981), Bourgain and Rosenthal [20] (1983), Ghoussoub and Rosenthal [44] (1983), and Rosenthal [126, 127, 128] (1981–1984). In [44] the so-called norm-sign-preserving operators on L_1 were considered, which are exactly the nonnarrow operators. There are also two citations that have an essential influence on the theory of narrow operators, even though they do not explicitly mention narrow operators: the book by Johnson, Maurey, Schechtman and Tzafriri [49], and Talagrand's paper [138].

The first systematic study of narrow operators was conducted by Plichko and Popov in the memoir [110] mentioned above. In 1996 V. Kadets and Popov [57] extended the notion of narrow operators to operators on $C(K)$-spaces. In 2001 V. Kadets, Shvidkoy and Werner [63] introduced another notion of narrow operators with domains equal to Banach spaces with the Daugavet property. In 2005 V. Kadets, Kalton and Werner [53] introduced hereditarily narrow operators. In 2009 O. Maslyuchenko, Mykhaylyuk and Popov [93] extended the definition of narrow operators to operators defined on vector lattices.

This book describes the current theory of narrow operators defined on function spaces and vector lattices. We aim to give a comprehensive presentation of known results and to include a complete bibliography. The only topic that we do not describe in detail are the operators introduced by V. Kadets, Shvidkoy and Werner [63], which are also called narrow operators, but which are very different from our narrow operators. Their theory is based on a completely different idea and is actually a part of the modern theory of Banach spaces with the Daugavet property; we suggest that this class of operators should be named Daugavet-narrow. These operators are of great interest, but they deserve a monograph of their own, and there is not enough space here to present all necessary background information for their study. We briefly mention them in Section 11.3 without giving any details, but do provide the relevant bibliography.

Chapter 1 contains preliminaries on F-spaces, Köthe function spaces, operator theory and vector lattices, including the definition and initial properties of narrow operators defined on a Köthe function F-space on an atomless measure space (Ω, Σ, μ).

In Chapter 2 we show that the class of narrow operators contains compact and AM-compact operators, Dunford–Pettis operators, operators whose ranges have smaller density than the domain space, and some other classes. We also show that every narrow operator from E to a Banach space can be restricted to a suitable subspace isometrically isomorphic to E, in such a way that the restriction is compact and has an arbitrarily small norm.

It turns out that for a large class of strictly nonconvex Köthe function F-spaces E including $L_p(\mu)$ with $0 < p < 1$, the only narrow operator defined on E is zero. Using this fact, in Chapter 3 we show that a homogeneous nonseparable $L_p(\mu)$-space, with $0 < p < 1$, has no nontrivial separable quotient space; we also give an elegant isomorphic classification of a class of spaces, which we call strictly nonconvex Köthe function F-spaces.

Chapter 4 is devoted to an example of a narrow projection of a rearrangement invariant (r.i.) space E onto a subspace isomorphic to E, showing that the class of narrow operators is not contained in any other class of "small" operators, including compact operators and strictly singular operators. In the separable case, this projection is described as the integration operator with respect to one variable acting on functions of two variables. This operator also plays an important role in other counterexamples.

In Chapter 5 we deal with the following natural questions about narrow operators: What subsets of the complex plane could be spectra of narrow operators? Is the conjugate operator of a narrow operator, narrow? Is the sum of two narrow operators narrow? Does the set of all narrow operators have the right-ideal property? Do numerical radii of narrow operators approximate the numerical index of L_p?

It is well known that L_1 has the Daugavet property, that is, the Daugavet equation $\|I + K\| = 1 + \|K\|$ is satisfied for every weakly compact operator K on L_1 where I is the identity of L_1. In Chapter 6 we show that the Daugavet property and some of its generalizations hold for narrow operators, and present applications to the geometric structure of $L_p(\mu)$-spaces. In particular, for each $1 \le p < \infty$, $p \ne 2$, there is a constant $k_p > 1$ such that if X is a complemented subspace of L_p and the projection P from L_p onto X satisfies $\|I - P\| < k_p$ then X is isomorphic to L_p. Further, if the Banach–Mazur distance between two spaces $L_p(\mu_i)$, $i = 1, 2$, is less that k_p then the corresponding measure spaces have isomorphic homogeneous parts are isomorphic, up to constant multiples, for details see Section 6.4.

We showed in Chapter 4 that narrowness does not imply strict singularity. In Chapter 7 we study in what situations various versions of strict singularity imply narrowness. This chapter contains some of the deepest results of this book. Many of them were obtained before the notion of narrowness was formally defined. We present the theorem of Bourgain and Rosenthal [20] that every ℓ_1-strictly singular operator from L_1 to a Banach space X is narrow and the very deep Rosenthal's characterization of narrow operators on L_1 [128], which in particular implies that every L_1-strictly singular operator on L_1, also called a non-Enflo operator on L_1, is narrow. We present this result together with its connections with pseudo-embeddings, pseudonarrow operators and the Enflo–Starbird maximal function λ. This combines results of Enflo and Starbird [37], Kalton [66] and Rosenthal [128]. We also present the theorem of Johnson, Maurey, Schechtman and Tzafriri's that every L_p-strictly singular operator on L_p, i.e. every non-Enflo operator on L_p, is narrow. We finish the chapter with some applications of these results. The study of ℓ_2-strictly singular operators logically belongs in this chapter, but the two known partial results require additional techniques so we present them in Chapters 9 and 10, respectively.

In Chapter 8 we discuss different notions of "weak" embeddings of L_1, namely semi-embeddings, G_δ-embeddings and sign-embeddings. We also present Talagrand's [138] construction of a subspace X of L_1 such that L_1 does not isomorphically embed in either X or L_1/X. This is interesting for us, because the corresponding quotient map is a nonnarrow operator from L_1 to a Banach space that contains no isomorphic copy of L_1. Thus, an L_1-strictly singular operator defined on L_1, that is, a non-Enflo operator defined on L_1, does not have to be narrow if the range space is an arbitrary Banach space.

Chapter 9 contains all known information concerning the Banach spaces X for which every operator from L_p to X is narrow. Here two facts should be mentioned. Every operator from L_p to L_r is narrow if $1 \leq p < 2$ and $p < r < \infty$, and this is no longer true for any other values of p and r. The second result asserts that every operator from L_p to ℓ_r is narrow if $r \neq 2$. The techniques developed in this chapter, which are quite interesting and include a probabilistic approach, allow us to prove a partial result concerning narrowness of ℓ_2-strictly singular operators, which is presented in Section 9.5.

One of the most striking facts concerning narrow operators is that, if an r.i. function space E has an unconditional basis then every operator on E is a sum of two narrow operators. In contrast, the sum of two narrow operators on L_1 is narrow. These phenomena are explained through the extension of the notion of narrow operators to vector lattices. O. Maslyuchenko, Mykhaylyuk and Popov [93] (2009) proved that the set of all narrow regular operators (i.e. differences of positive operators) between lattices that are "nice enough," including L_p, form a band, and so, in particular a sum of two narrow regular operators is narrow, like for operators on L_1. In fact all operators on L_1 are regular, so the phenomenon of sums on L_1 is a special case of the general behavior of regular narrow operators on "nice" vector lattices. In Chapter 10 we present a generalization of Kalton's and Rosenthal's representation theorems for operators on L_1 to vector lattices, which was proved in [93]. The last Section 10.9 contains a generalization of a result of Flores and Ruiz [39] about narrowness of regular ℓ_2-strictly singular operators.

Chapter 11 contains some variants of the notion of narrow operators. One of them, hereditarily narrow operators, allowed V. Kadets, Kalton and Werner [53] to prove the strongest generalization of Pełczyński's theorem on the impossibility of the isomorphic embedding of L_1 into a Banach space with an unconditional basis. Another variant, gentle narrow operators introduced in [102], is used to give a partial answer to the problem whether Rosenthal's characterization of narrow operators on L_1 can be generalized to L_p for $1 \leq p < 2$. Next we present the notion of C-narrow operators on $C(K)$-spaces defined in [57]. Since $C(K)$-spaces do not contain characteristic functions, the definition of C-narrow operators is based on Rosenthal's characterization of narrow operators on L_1. This approach proved quite fruitful and C-narrow operators share many properties of narrow operators on Köthe–Banach spaces. The last two sections are devoted to the usual notion of narrow operators but in somewhat unusual settings. Most of the results on narrow operators use the absolute continuity of the norm of the domain. Investigation of narrow operators defined on spaces without this property, like L_∞, leads to many surprising results. For example, there exist compact operators and even linear functionals on L_∞ that are not narrow. We present the known results and open problems in this setting in Section 11.4. We finish the chapter with a result that every 2-homogeneous scalar polynomial on L_p, $1 \leq p < 2$, is narrow. We think that it would be interesting to investigate the notion of narrowness for polynomials on Banach spaces.

A number of proofs in this book are new, and some of them are due to our colleagues. Whenever we present their proofs, we gratefully credit the authors.

The concept of narrow operators, a subject of numerous investigations during the last 30 years, gave rise to a number of attractive open problems. We state these problems throughout this book near the context from which they originate. For the convenience of the reader, in the last Chapter 12 we list all open problems that were stated in different chapters. We hope that the book will inspire new work on these problems.

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Chernivtsi/Oxford (Ohio),
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Mikhail Popov,
Beata Randrianantoanina

Contents

Chapter 1

Introduction and preliminaries

1.1 Background information

We assume that our readers are familiar with functional analysis in general, and the well-known facts on Banach spaces. Standard texts on this material include books by Lindenstrauss and Tzafriri [79, 80] and Albiac-Kalton [3]. Some sections require elementary knowledge on vector lattices; we refer the reader to the books by Aliprantis-Burkinshaw [6] and Kusraev [74].

For the sake of generality, in a number of statements we consider the so-called Köthe F-spaces as domain spaces for narrow operators. We understand that some of the readers may not be interested in the non-Banach case. However, some significant applications to the isomorphic structure concern the nonlocally convex F-spaces, (these are contained in Chapter 3, and do not play a significant role in other chapters). Our main reference in this direction is Rolewicz's monograph [122].

1.2 Terminology and notation

Throughout the book, we consider Köthe function spaces (for definitions see below) on finite atomless measure spaces only. We concentrate on atomless spaces since, by definition, a narrow operator must send any atom to zero, and hence a narrow operator is a direct sum of the zero operator on the atomic part and a narrow operator on the atomless part of a measure space. Furthermore, an operator defined on a Köthe F-space on an infinite atomless measure space is narrow if and only if all its restrictions to finite measure parts are narrow. Therefore, we consider finite atomless measure spaces only.

Let (Ω, Σ, μ) be a finite atomless measure space. For $A \in \Sigma$, we set $\Sigma(A) = \{B \in \Sigma \colon B \subseteq A\}$ and $\Sigma^+(A) = \{B \in \Sigma(A) \colon \mu(B) > 0\}$. In particular, $\Sigma^+ = \Sigma^+(\Omega)$. By a sub-$\sigma$-algebra of Σ we mean a sigma-algebra Σ_1 of subsets of Ω such that $\Sigma_1 \subseteq \Sigma$. In particular, $\Omega \in \Sigma_1$ for each sub-σ-algebra Σ_1 of Σ. For A, $A_i \in \Sigma$, $i \in I$ the notation $A = \bigsqcup_{i \in I} A_i$ means that $A = \bigcup_{i \in I} A_i$ and simultaneously $A_i \cap A_j = \emptyset$ for each $i, j \in I$, $i \neq j$. For the σ-algebra of Lebesgue measurable subsets of the unit segment $[0, 1]$ we use the same notations Σ and μ as in the case of an arbitrary measure space (Ω, Σ, μ). By $\mathbf{1}_A$ we denote the characteristic function of a set $A \in \Sigma$.

If M is a subset of a vector space X, then $\operatorname{lin} M$ denotes the linear span of M in X, that is, $\operatorname{lin} M = \{\sum_{k=1}^{n} a_k x_k \colon n \in \mathbb{N},\, a_k \in \mathbb{K},\, x_k \in M,\, k = 1, \dots, n\}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a scalar field. The equality $X = Y \oplus Z$ for a linear space X and its subspaces Y, Z means that X is a direct sum of subspaces Y and Z, that is, for every $x \in X$ there exist unique elements $y \in Y$ and $z \in Z$, such that $x = y + z$.

For a Banach space X by B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. The symbol $[x_n]_{n=1}^\infty$ (or simply $[x_n]$) denotes the closed linear span of a sequence $(x_n)_{n=1}^\infty$; and $\overline{M}$ denotes the closure of a subset M of X.

Let $1 \le p \le \infty$ and let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces. By $(\sum_{n=1}^\infty X_n)_p$ we mean the ℓ_p-sum of X_ns, that is, the Banach space of all sequences $x = (x_n)_{n=1}^\infty$, $x_n \in X_n$ such that $\|x\| \stackrel{\text{def}}{=} (\sum_{n=1}^\infty \|x_n\|^p)^{1/p} < \infty$ if $p < \infty$ and for $p = \infty$ we set $\|x\| \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \|x_n\| < \infty$.

For Banach or F-spaces X and Y, the symbol $\mathcal{L}(X, Y)$ stands for the linear space of all continuous linear operators from X to Y. In the case when X, Y are Banach spaces, $\mathcal{L}(X, Y)$ is a Banach space endowed with the standard norm. If $X = Y$ then we shorten $\mathcal{L}(X) = \mathcal{L}(X, X)$.

1.3 Narrow operators on function spaces

F-spaces

An *F-space* is a complete metric linear space X over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with an invariant metric ρ (i.e. $\rho(x, y) = \rho(x + z, y + z)$ for each $x, y, z \in X$). Any complete metric linear space has an equivalent invariant metric [122, Theorem 1.1.1]. We set $\|x\| = \rho(x, 0)$, and so, $\rho(x, y) = \|x - y\|$, because ρ is invariant. Thus defined map $\|\cdot\| : X \times X \to [0, +\infty)$ is called the F-norm of the F-space X.

A very important class of F-spaces that are not Banach spaces is the class of $L_p(\mu)$-spaces with $0 \le p < 1$. Given a finite measure space (Ω, Σ, μ) and $0 < p < 1$, $L_p(\mu)$ is defined as the linear space (with respect to the natural operations of addition and multiplication by scalars) of all classes of equivalent Σ-measurable functions $x : \Omega \to \mathbb{K}$ such that $\|x\|_p \stackrel{\text{def}}{=} \int_\Omega |x|^p \, d\mu < \infty$. The space $L_0(\mu)$ is defined as the linear space of all classes of equivalent Σ-measurable functions $x : \Omega \to \mathbb{K}$ with the metric $\|x\|_0 \stackrel{\text{def}}{=} \int_\Omega |x|/(1 + |x|) \, d\mu < \infty$.

The space $L_p(\mu)$ for $0 \le p < 1$ is an F-space with respect to the defined above F-norm. It is well known and not hard to check that the metric convergence in $L_0(\mu)$ is equivalent to the convergence in measure.

For elements x, y of $L_0(\mu)$ the inequality $x \le y$ means that $x(\omega) \le y(\omega)$ for almost all $\omega \in \Omega$.

An important class of Banach spaces, which are linear subspaces of $L_0(\mu)$, are the spaces $L_p(\mu)$ for $1 \le p < \infty$ defined by

$$L_p(\mu) = \left\{ x \in L_0(\mu) \colon \|x\| = \left(\int_\Omega |x|^p \, d\mu \right)^{1/p} < \infty \right\},$$

and the space

$$L_\infty(\mu) = \left\{ x \in L_0(\mu) \colon \|x\| = \inf_{A \in \Sigma,\, \mu(A)=0} \sup_{t \in \Omega \setminus A} |x(t)| < \infty \right\} .$$

Köthe F-spaces and rearrangement-invariant F-spaces

Let (Ω, Σ, μ) be a finite measure space. An F-space E of equivalence classes of measurable functions on Ω is called a *Köthe F-space* if the following conditions hold:

(K_i) if $y \in E$ and $|x| \leq |y|$ then $x \in E$ and $\|x\| \leq \|y\|$;

(K_{ii}) $\mathbf{1}_\Omega \in E$.

If, moreover, E is a Banach space and

(K_{iii}) $E \subseteq L_1(\mu)$

then E is called a *Köthe–Banach space*.

Note that, in the terminology of Lindenstrauss–Tzafriri [80, p. 28], a Köthe function space is a Köthe–Banach F-space in our terminology, without the assumption of integrability of its elements.

Let E be a Köthe–Banach space. Using the closed graph theorem, one can show that, by (K_{iii}), the inclusion embedding of E to $L_1(\mu)$ is continuous. For the same reason, every element $y \in L_1(\mu)$ such that $x \cdot y \in L_1(\mu)$ for every $x \in E$, defines an element $g \in E^*$ by

$$g(x) = \int_\Omega xy \, \mathrm{d}\mu \, . \tag{1.1}$$

The set of all elements of E^* of form (1.1) is denoted by E'. It is a Köthe–Banach space with respect to the norm $\|y\|_{E'} \stackrel{\text{def}}{=} \sup\{\int_{\Omega'} xy \, \mathrm{d}\mu' : x \in B_E\}$.

A Köthe F-space E is said to have an *absolutely continuous norm* provided that $\lim_{n\to\infty} \|x \cdot \mathbf{1}_{A_n}\| = 0$ for each $x \in E$ and every decreasing sequence of sets $A_n \in \Sigma$ with $\bigcap_{n=1}^\infty A_n = \emptyset$. It is not hard to show that, if E is a Köthe F-space on a finite measure space (which is always assumed in this book) then the norm of E is absolutely continuous if and only if $\lim_{\mu(A)\to 0} \|x \cdot \mathbf{1}_A\| = 0$ for each $x \in E$.

The majority of results on narrow operators are obtained under the assumption that the norm of the domain space is absolutely continuous. However, it is possible to replace this assumption with a weaker property, which is sufficient for most of these proofs.

Definition 1.1. We say that a Köthe F-space E on a finite atomless measure space has an *absolutely continuous norm on the unit* if $\lim_{\mu(A)\to 0} \|\mathbf{1}_A\| = 0$.

Observe that if a Köthe F-space E on (Ω, Σ, μ) has an absolutely continuous norm on the unit, then E has an absolutely continuous norm on "any essentially bounded element.” Indeed, let $x \in L_\infty$, say, $\|x\|_{L_\infty} \le n_0$ for some $n_0 \in \mathbb{N}$. Then for each $A \in \Sigma$ we have $\|x \cdot \mathbf{1}_A\| \le \|n_0 \mathbf{1}_A\| \le n_0 \|\mathbf{1}_A\| \to 0$ as $\mu(A) \to 0$.

Evidently, an absolutely continuous norm is an absolutely continuous norm on the unit. The following example shows that the converse is not true.

Example 1.2. There exists a Köthe–Banach space E on $[0, 1]$ with an absolutely continuous norm on the unit, which is not absolutely continuous.

Proof. Let E_0 be any Köthe–Banach space on $[0, 1]$ with an absolutely continuous norm, and let $[0, 1] = \bigsqcup_{n=1}^\infty A_n$ with $A_n \in \Sigma^+$ for each $n \in \mathbb{N}$. Then the Köthe–Banach space $E = (\bigcup_{n=1}^\infty E_0(A_n))_\infty$ has the desired properties. Indeed, since $\|\mathbf{1}_A\|_E = \sup_n \|\mathbf{1}_{A \cap A_n}\|_{E_0} \le \|\mathbf{1}_A\|_{E_0}$, we have that $\lim_{\mu(A) \to 0} \|\mathbf{1}_A\|_E = 0$ by the absolute continuity of the norm of E_0. Thus, setting $x = \sum_{n=1}^\infty \|\mathbf{1}_{A_n}\|_{E_0}^{-1} \mathbf{1}_{A_n}$, we obtain that $\|x \cdot \mathbf{1}_{A_n}\|_E = 1$ for each $n \in \mathbb{N}$, and hence, the norm of E is not absolutely continuous. □

The space $L_p(\mu)$ has an absolutely continuous norm if $0 \le p < \infty$ and does not have an absolutely continuous norm on the unit if $p = \infty$.

For a function $f \in L_0(\mu)$, we define the *distribution function* $d_f \colon \mathbb{R} \to [0, \mu(\Omega)]$ of f by $d_f(a) = \mu\{\omega \in \Omega : f(\omega) > a\}$.

We say that two functions f and g are *equimeasurable* if they have the same distribution function, that is, $d_f(a) = d_g(a)$ for all $a \in \mathbb{R}$. Two functions f and g are called *equimeasurable in modulus* if their moduli have the same distribution function, that is, $d_{|f|}(a) = d_{|g|}(a)$ for all $a \ge 0$.

Following Braverman [22], we will say that an F-space E on a finite measure space (Ω, Σ, μ) is *rearrangement-invariant* (r.i.) if

(a) if $x \in E$ and $|y| \le |x|$ a.e. then $y \in E$ and $\|y\|_E \le \|x\|_E$, and

(b) if $x \in E$ and $d_{|y|} = d_{|x|}$ then $y \in E$ and $\|y\|_E = \|x\|_E$.

We note that in the literature, some authors include additional conditions in the definition of an r.i. space. Some authors also use a term “symmetric space” in this context.

We will always consider r.i. spaces on a finite atomless measure space (Ω, Σ, μ). All the spaces $L_p(\mu)$ with $0 \le p \le \infty$ are r.i. F-spaces. For more information on r.i. F-spaces see [22, 80].

Some important systems of functions

We define the *dyadic intervals* as $I_n^k = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$.

The L_∞-normalized *Haar system* is the following sequence in L_∞: $\overline{h}_1 = \mathbf{1}$ and

$$\overline{h}_{2^n+k} = \mathbf{1}_{I_{n+1}^{2k-1}} - \mathbf{1}_{I_{n+1}^{2k}}$$

for $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$. Sometimes it is convenient to enumerate the Haar system in a different way, namely $\overline{h}_{0,0} = \mathbf{1}$, $\overline{h}_{n,k} = \mathbf{1}_{I_{n+1}^{2k-1}} - \mathbf{1}_{I_{n+1}^{2k}}$ for $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$. We reserve the notation $(h_n)_{n=1}^\infty$ or $(h_{0,0}) \cup (h_{n,k})_{n=0}^\infty{}_{k=1}^{2^n}$ for the normalized Haar system in a Köthe space E on [0, 1].

It is well known that the Haar system is a monotone basis of every separable r.i. Banach space E on [0, 1] [80, p. 150], and it is an unconditional basis of $E = L_p$ for any $p \in (1, \infty)$ [80, p. 155]. Moreover, it is an unconditional basis of a separable r.i. space E on [0, 1] if and only if the Boyd indices of E satisfy $1 < p_E$ and $q_E < \infty$ [80, p. 157]. The Haar system is the "best" possible system in any r.i. space E in the following sense: if E embeds isometrically into a Banach space X with a basis (x_n) then, for each $\varepsilon > 0$ there exists a block basis (u_k) of (x_n), which is $(1+\varepsilon)$-equivalent to the Haar system in E (this property is called the *precise reproducibility* of the Haar system [80, p. 158]). In particular, it follows that if E embeds isomorphically into a Banach space with an unconditional basis then the Haar system is unconditional in E, and in this case the unconditional constant of the Haar system is the least possible among all unconditional constants of unconditional bases of E.

Definition 1.3. Let (Ω, Σ, μ) be a finite atomless measure space. A sequence of sets $(G_{n,k})_{n=0,k=1}^{\infty \ \ 2^n}$, $G_{n,k} \in \Sigma$ is called a *tree of sets* if

$$G_{n,k} = G_{n+1,2k-1} \sqcup G_{n+1,2k} \text{ and } \mu(G_{n+1,2k-1}) = \mu(G_{n+1,2k}) = \tfrac{1}{2}\, \mu(G_{n,k}),$$

for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$. The corresponding system of functions $(g_i)_{i=1}^\infty$ defined by $g_1 = \mathbf{1}_{G_{0,1}}$ and $g_{2^n+k} = \mathbf{1}_{G_{n+1,2k-1}} - \mathbf{1}_{G_{n+1,2k}}$ for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$ is called a *Haar-type system.*

As in the case of the Haar system, we will also sometimes use another notation for a Haar-type system: $(g_{0,0}) \cup (g_{k,i})_{n=0,k=1}^{\infty \ \ 2^n}$ and for a tree of sets (see Definition 7.5).

The following notion plays an important role in the book.

Definition 1.4. An element $x \in L_0(\mu)$ is called a *sign* if x takes values in the set $\{-1, 0, 1\}$, and a *sign on* $A \in \Sigma$, if it is a sign with $\operatorname{supp} x = A$. A sign x is *of mean zero*, provided $\int_\Omega x \, \mathrm{d}\mu = 0$.

Let (Ω, Σ, μ) be a finite atomless measure space. Given $A \in \Sigma^+$, by a *Rademacher system* on A we mean any sequence of mean zero signs on A which are independent random variables. In other words, a sequence (r_n) of mean zero signs on A is a Rademacher system on A if for any $n \in \mathbb{N}$ and any n-tuple of sign numbers $(\theta_k)_{k=1}^n$, $\theta_k = \pm 1$ we have $\mu\{\omega \in \Omega : r_k(\omega) = \theta_k,\ k = 1, \dots, n\} = 2^{-n} \mu(A)$. Since μ is atomless, it is easy to construct such a sequence. The usual Rademacher system on [0, 1], or just the *Rademacher system*, is the sequence defined using the Haar system by

$$r_n = \sum_{k=2^{n-1}+1}^{2^n} \overline{h}_k = \sum_{k=1}^{2^{n-1}} \overline{h}_{n-1,k}, \quad n = 1, 2, \dots .$$

Note that $r_n(t) = \operatorname{sign}\sin(2^n \pi t)$ for each $n \in \mathbb{N}$ and almost all $t \in [0,1]$. If an r.i. space E on $[0,1]$ is not equal to L_∞, up to an equivalent norm, then (r_n) is a weakly null sequence in E [80, p. 160].

By $\mathbb{N}^{<\omega}$ we denote the set of all finite subsets of the positive integers. The *Walsh system* $(w_I)_{I \in \mathbb{N}^{<\omega}}$ is defined as

$$w_I = \prod_{i \in I} r_i \ ,$$

where (r_n) is the Rademacher system (in particular, $w_\emptyset = \mathbf{1}$, by convention). The Walsh system with respect to the lexicographical order $w_\emptyset$, $w_{\{1\}}$, $w_{\{2\}}$, $w_{\{1,2\}}$, $w_{\{3\}}$, $w_{\{1,3\}}$, $w_{\{2,3\}}$, $w_{\{1,2,3\}}, \ldots$ is a Schauder basis of L_p for $1 < p < \infty$. The Walsh system is an orthonormal basis of L_2, a conditional basis of L_p for $p \neq 2$, and a Markushevich basis of L_1.

Conditional expectation operator

Let (Ω, Σ, μ) be a measure space and let $\mathcal{F}$ be a sub-σ-algebra of Σ. The *conditional expectation* $y = M^{\mathcal{F}} x$ of an $x \in L_1$ (or, generally, $x \in L_1(X)$ where X is a Banach space) with respect to $\mathcal{F}$ is defined as the $\mathcal{F}$-measurable function $y \in L_1(\mathcal{F})$ (respectively, $y \in L_1(X)$) such that $\int_A x \, \mathrm{d}\mu = \int_A y \, \mathrm{d}\mu$, for each $A \in \mathcal{F}$. The uniqueness of the conditional expectation is obvious; its existence in the scalar case is guaranteed by the classical Radon–Nikodým theorem; in the vector-valued case is proved in [29, p. 123], and in the case when $\mathcal{F}$ is generated by a finite partition $\Omega = \bigsqcup_{k=1}^m A_k$, $A_k \in \Sigma$, is easily verified. Indeed, for each $x \in L_1(X)$ we have

$$M^{\mathcal{F}} x = \sum_{k=1}^m \Big(\frac{1}{\mu(A_k)} \int_{A_k} x \, \mathrm{d}\mu \Big) \mathbf{1}_{A_k} \ . \tag{1.2}$$

It is well known that the conditional expectation operator $M^{\mathcal{F}}$ with respect to any sub-σ-algebra $\mathcal{F}$ of Σ is a contractive projection on $L_p(X)$, for any $p \in [1, +\infty)$ [29, p. 122].

A typical example of a conditional expectation operator with respect to a nonatomic sub-σ-algebra is the integration in $L_1[0,1]^2$ over one of the variables

$$(M^{\mathcal{F}} x)(s,t) = \int_{[0,1]} x(s,t') \, \mathrm{d}t' \ ,$$

where $\mathcal{F} = \{A \times [0,1] : A \in \Sigma\}$.

Narrow operators and rich subspaces of Köthe spaces

For the rest of the book, unless specifically noted, by a Köthe F-space we mean a Köthe F-space on a finite atomless measure space.

Definition 1.5. Let E be a Köthe F-space and X be an F-space. An operator $T \in \mathcal{L}(E, X)$ is called *narrow* if for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a mean zero sign x on A such that $\|Tx\| < \varepsilon$. If for each $A \in \Sigma^+$ there exists a mean zero sign x on A such that $Tx = 0$ then T is called *strictly narrow.*

The property for an operator to be strictly narrow is a property of its kernel. This leads us to the notion of a strictly rich subspace.

Definition 1.6. Let E be a Köthe F-space. A subspace X of E is called *strictly rich* if for every $A \in \Sigma^+$ there exists a mean zero sign x on A that belongs to X.

Obviously, X is strictly rich if and only if the quotient map from E onto E/X is strictly narrow. On the other hand, an operator is strictly narrow if and only if its kernel is strictly rich.

Definition 1.7. Let E be a Köthe F-space. A subspace X of E is called *rich* if the quotient map from E onto E/X is narrow.

In other words, X is rich if for every $A \in \Sigma^+$ and every $\varepsilon > 0$ there exists a mean zero sign y on A and $x \in X$ such that $\|x - y\| < \varepsilon$.

The same definitions can be applied for a wider class of maps. In particular, to prove that every order-to-norm continuous AM-compact operator (see Section 10.2) from a vector lattice to a Banach space is narrow, we consider nonlinear narrow maps.

The following left-ideal property of narrow operators immediately follows from the definition.

Proposition 1.8. *Let E be a Köthe F-space and let X, Y be F-spaces. If $T : E \to X$ is a narrow (resp., strictly narrow) continuous linear operator and $S : X \to Y$ is a continuous linear operator then $ST : E \to Y$ is narrow (resp., strictly narrow).*

However, the set of all narrow operators from a Köthe F-space E to an F-space X, as we will see later, very seldom forms a linear subspace in the space $\mathcal{L}(E, X)$ (see Chapter 5).

If the domain space has an absolutely continuous norm on the unit then the condition on signs to be of mean zero can be equivalently removed from Definition 1.5. Moreover, the following conditions are sufficient for an operator to be narrow.

Proposition 1.9. *Let E be a Köthe F-space with an absolutely continuous norm on the unit, X an F-space and $T \in \mathcal{L}(E, X)$. Then the following assertions are equivalent:*

(i) *T is narrow.*

(ii) *For each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a sign x on A such that $\|Tx\| < \varepsilon$.*

(iii) *For each $A \in \Sigma$ and each $\varepsilon > 0$ there are $B \in \Sigma(A)$ and a sign x on B such that $\mu(B) \geq \mu(A)/2$ and $\|Tx\| < \varepsilon$.*

Proof. (iii) $\Rightarrow$ (ii). Fix $A \in \Sigma$ and $\varepsilon > 0$. Choose $B_1 \in \Sigma(A)$ and a sign x_1 on B_1 so that $\mu(B_1) \geq \mu(A)/2$ and $\|Tx_1\| < \varepsilon/2$. Then set $A_1 = A \setminus B_1$, and choose $B_2 \in \Sigma(A_1)$ and a sign x_2 on B_2 so that $\mu(B_2) \geq \mu(A_1)/2 \geq \mu(A)/4$ and $\|Tx_2\| < \varepsilon/4$. Suppose $B_1, \dots, B_n$ and $x_1, \dots, x_n$ have been chosen. Put $A_n = A \setminus (B_1 \cup \ldots \cup B_n)$ and choose $B_{n+1} \in \Sigma(A_n)$ and a sign x_{n+1} on B_{n+1} so that $\mu(B_{n+1}) \geq \mu(A_n)/2$ and $\|Tx_{n+1}\| < \varepsilon/2^{n+1}$. Then put $x = \sum_{n=1}^{\infty} x_n$ (the series converges because E has an absolutely continuous norm on the unit). Since (x_n) are disjoint signs and $\mu(A \setminus \bigcup_{n=1}^{\infty} B_n) = 0$, x is a sign on A and $\|Tx\| < \varepsilon$ by the construction.

(ii) $\Rightarrow$ (i). Fix $A \in \Sigma$ and $\varepsilon > 0$. The absolute continuity of the norm on the unit yields that there is $\delta > 0$ such that for every $B \in \Sigma$ the condition $\mu(B) < \delta/2$ implies $\|2 \cdot \mathbf{1}_B\| < \varepsilon/(2\|T\|)$ (for general F-spaces we cannot take a constant out of the norm, however $\|\alpha u_n\| \to 0$ as $\|u_n\| \to 0$ is still true). Now we choose $n \in \mathbb{N}$ so that $\mu(A)/n < \delta$ and partition A into n subsets $A_1, \dots, A_n$ of measure $\mu(A)/n$. For each $k = 1, \dots, n$ choose a sign x_k on A_k with $\|Tx_k\| < \varepsilon/(2n)$ and set $a_k = \int_\Omega x_k \, \mathrm{d}\mu$. Since $|a_k| \leq \mu(A)/n$ for all k, we can inductively choose sign numbers $\theta_k = \pm 1$ so that

$$\Big| \sum_{k=1}^{n} \theta_k a_k \Big| \leq \frac{\mu(A)}{n} .$$

Then $\overline{x} = \sum_{k=1}^{n} \theta_k x_k$ is a sign on A with $\|T\overline{x}\| < \varepsilon/2$ and

$$\Big| \int_\Omega x_k \, \mathrm{d}\mu \Big| \leq \frac{\mu(A)}{n} .$$

Now we put $A^+ = \{\omega \in \Omega : \overline{x}(\omega) = 1\}$ and $A^- = A \setminus A^+$. Observe that $\overline{x} = \mathbf{1}_{A^+} - \mathbf{1}_{A^-}$ and $|\mu(A^+) - \mu(A^-)| \leq \mu(A)/n$. Without loss of generality we may and do assume that $\mu(A^+) \geq \mu(A^-)$. Choose $A_0 \in \Sigma(A^+)$ so that $2\mu(A_0) = \mu(A^+) - \mu(A^-)$ and set

$$x(\omega) = \begin{cases} -1 & \text{if } \omega \in A_0, \\ \overline{x}(\omega) & \text{if } \omega \in \Omega \setminus A_0. \end{cases}$$

It remains to observe that x is a mean zero sign on A and

$$\|Tx\| \leq \big\|T(x - \overline{x})\big\| + \|T\overline{x}\| < \|T\| \big\|2 \cdot \mathbf{1}_{A_0}\big\| + \frac{\varepsilon}{2} < \varepsilon .$$

The implication (i) $\Rightarrow$ (iii) is obvious. □

We do not know whether Definition 1.5 remains the same if for operators from $\mathcal{L}(L_\infty, X)$, the condition on a sign to be of mean zero, is omitted.

Open problem 1.10. Does Definition 1.5 remain the same for $E = L_\infty$ if the condition on a sign to be of mean zero is omitted?

Theorem 11.55 below provides a partial answer to this problem.

The following lemma is a useful characterization of narrow operators.

Lemma 1.11. *Let E be a Köthe F-space, X an F-space, and $T \in \mathcal{L}(E, X)$ a narrow operator. Then for each $A \in \Sigma$, $\varepsilon > 0$ and any integer $n \geq 1$ there exists a partition $A = A' \sqcup A''$ into disjoint subsets of measures $\mu(A') = (1-2^{-n})\mu(A)$ and $\mu(A'') = 2^{-n}\mu(A)$ such that $\|Th\| < \varepsilon$, where $h = \mathbf{1}_{A'} - (2^n - 1)\mathbf{1}_{A''}$.*

For the proof one should use the definition n times.

We will also use the following elementary fact.

Lemma 1.12. *Let E be a Köthe–Banach space on $[0, 1]$ with an absolutely continuous norm on the unit, X a Banach space and $T \in \mathcal{L}(E, X)$. Assume that for any dyadic interval I_n^k and any $\varepsilon > 0$, there exists $x \in E$ such that $x^2 = \mathbf{1}_{I_n^k}$ and $\|Tx\| < \varepsilon$. Then T is narrow.*

Proof. Given any $A \in \Sigma$ and $\varepsilon > 0$, we choose $n \in \mathbb{N}$ and $J \subseteq \{1, \dots, 2^n\}$ so that for $I = \bigcup_{j \in J} I_n^j$ we have $\|\mathbf{1}_{A \triangle I}\| < \frac{\varepsilon}{2\|T\|}$. Then for each $j \in J$ we find $x_j \in E$ such that $x_j^2 = \mathbf{1}_{I_n^j}$ and $\|Tx_j\| < \frac{\varepsilon}{2m}$ where $m = |J|$. Then setting $\overline{x} = \sum_{j \in J} x_j$ we have that $x^2 = \mathbf{1}_I$ and $\|T\overline{x}\| \leq \sum_{j \in J} \|Tx_j\| < \varepsilon/2$.

Finally, setting $x = \mathbf{1}_{A \setminus I} + \overline{x} \cdot \mathbf{1}_{A \cap I}$ we obtain $x^2 = \mathbf{1}_A$ and $\|x - \overline{x}\| = \|\mathbf{1}_{A \triangle I}\|$. Thus, by the above,

$$\|Tx\| \leq \|T\overline{x}\| + \|T\|\|x - \overline{x}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

By Proposition 1.9, T is narrow. □

1.4 Homogeneous measure spaces and Maharam's theorem

When considering an r.i. space E on a general measure space (Ω, Σ, μ), sometimes it is convenient to consider E over an isomorphic measure space of a special kind which is provided by the Maharam theorem, as we describe below.

Some set-theoretical terminology

For the set-theoretical terminology we refer the reader to Jech's book [48]. A set A is called *transitive* if the relations $C \in B \in A$ imply $C \in A$. A transitive set which is well ordered by the relation $\in$ is called an *ordinal*. For example, the sets

$$0 \stackrel{\text{def}}{=} \emptyset,\ 1 \stackrel{\text{def}}{=} \{0\} = \{\emptyset\},\ 2 \stackrel{\text{def}}{=} \{0, 1\} = \{\emptyset, \{\emptyset\}\},\ 3 \stackrel{\text{def}}{=} \{0, 1, 2\} = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\}, \dots$$

are ordinals. The set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is transitive but is not an ordinal. The axioms of ZFC (more precisely, the axioms of choice and regularity) imply that, if a transitive set is linearly ordered by the relation $\in$ then it is an ordinal. Any well-ordered set A is isomorphic to a unique ordinal γ (i.e. there exists a monotone bijection between A and γ). For any two ordinals β, γ one of the following mutually exclusive relation holds: $\beta \in \gamma$, $\beta = \gamma$, $\gamma \in \beta$. So, the natural well ordering of ordinals is defined as $\beta < \gamma$ if and only if $\beta \in \gamma$. In particular, an ordinal γ equals the set of all ordinals less than γ.

The cardinality of a set A is denoted by $|A|$. A *cardinal* is any ordinal γ having the property that $|\beta| < |\gamma|$ for every $\beta \in \gamma$. An infinite cardinal γ is denoted by ω_α, where α is the ordinal which is isomorphic to the set of all infinite cardinals, less than γ. For example, ω_0 is the least (countable) infinite cardinal; ω_1 is the least uncountable cardinal. For every ordinal α there exists the cardinal ω_α. The cardinality of the cardinal ω_α is denoted by $\aleph_\alpha$, i.e. $|\omega_\alpha| = \aleph_\alpha$.

Isomorphisms of measure spaces, homogeneous spaces

Given a measure space (Ω, Σ, μ), by $\widetilde{\Sigma}$ we denote the Boolean σ-algebra of sets from Σ, equal up to a set of measure zero, with respect to the natural operations $\vee$ and $\wedge$. Two measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ are called *isomorphic*, if there exists a Boolean measure-preserving isomorphism between the Boolean algebras $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$. We say that an isomorphism $I : (\widetilde{\Sigma}_1, \mu_1) \to (\widetilde{\Sigma}_2, \mu_2)$ *is induced by a map* $J : \Omega_1 \to \Omega_2$ provided that $I(\phi_1(A)) = \phi_2(J(A))$ for every $A \in \Sigma_1$, where ϕ_i is the quotient map from Σ_i to $\widetilde{\Sigma}_i$ for $i = 1, 2$.

Note that r.i. spaces defined on isomorphic measure spaces are isometric. Indeed, for every $f \in E(\Omega_1, \Sigma_1, \mu_1)$, if τ is an isomorphism between $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ then $Tf(t) = f(\tau^{-1}(t))$ defines an equimeasurable function on $E(\Omega_2, \Sigma_2, \mu_2)$.

The *density* $\operatorname{dens} X$ of a topological space X is defined to be the least cardinality of a dense subset of X. Let (Ω, Σ, μ) be a measure space. By $\operatorname{dens} \Sigma$ we denote the density of the metric space $(\widetilde{\Sigma}, \rho)$ where $\rho(A, B) = \mu(A \triangle B)$. One can show that $\operatorname{dens} \Sigma = \operatorname{dens} L_p(\Omega, \Sigma, \mu)$ for every $p \in [0, \infty)$.

A measure space (Ω, Σ, μ) is called *homogeneous* if $\operatorname{dens} \Sigma(A) = \operatorname{dens} \Sigma$ for each $A \in \Sigma^+$. By the Maharam theorem [87], [75, p. 122], [110, p. 17], a homogeneous measure space (Ω, Σ, μ) is isomorphic to the measure space $(D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$, where D^{ω_α} is the ω_αth power of the two-point set $D = \{-1, 1\}$ with the Haar measure μ_{ω_α} on the compact Abelian group D^{ω_α} where $\aleph_\alpha = \operatorname{dens} \Sigma$.

If the reader has never considered this measure space, it is worthwhile to outline some details of the construction. Given any nonempty set J, the set D^J is the set of all J-sequences of sign numbers $\xi : J \to D$. For any finite subset $F \subseteq J$ and a collection $\Theta = (\theta_j)_{j \in F}$ of sign numbers $\theta_j \in D$ we define the *cylindric set*

$$A_{F,\Theta} = \left\{\xi \in D^J : \xi(j) = \theta_j \ \text{ for each } \ j \in F\right\}$$

of measure $\mu_J(A_{F,\Theta}) = 2^{-|F|}$. The collection of all disjoint unions of cylindric sets is an algebra, called the *cylindric algebra*, to which the measure μ_J is extended in the obvious manner. Then Σ_J is defined to be the least σ-algebra containing the cylindric algebra, with the extended measure μ_J.

Definition 1.13. We say that a function $x : D^J \to \mathbb{K}$ *depends on the coordinate* $j \in J$ if the set $\{\xi \in D^{J\setminus\{j\}} : x(\xi \times \{1\}) \neq x(\xi \times \{-1\})\}$ is not a null set with respect to the measure $\mu_{J\setminus\{j\}}$.

A measurable function $x : D^{\omega_\alpha} \to \mathbb{K}$ depends on, at most, countable set of coordinates [36].

The Maharam theorem

By a *disjoint union* of sets $(A_i)_{i\in I}$ we mean a union of "disjoint copies" of these sets, for example $\bigcup_{i\in I}(A_i \times \{i\})$. Let $(\Omega_i, \Sigma_i, \mu_i)$, $i \in I$ be any family of measure spaces. Denote by Ω the disjoint union of the sets $\Omega = \bigcup_{i\in I} \Omega_i \times \{i\}$ and let

$$\widetilde{\Omega}_i = \Omega_i \times \{i\} \quad \text{and} \quad \widetilde{\Sigma}_i = \big\{B \times \{i\} : B \in \Sigma_i\big\}, \quad \text{for all} \quad i \in I .$$

Define a σ-algebra Σ of subsets of Ω as $\Sigma = \{A \subseteq \Omega : (\forall i \in I)\,(A \cap \widetilde{\Omega}_i \in \widetilde{\Sigma}_i)\}$, and a measure μ on Σ by $\mu(A) = \sum_{i\in I} \mu_i(A \cap \widetilde{\Omega}_i)$.

The measure space (Ω, Σ, μ) is called the *direct sum of measure spaces* $(\Omega_i, \Sigma_i, \mu_i)$, $i \in I$ and is denoted by

$$(\Omega, \Sigma, \mu) = \sum_{i\in I} \oplus\, (\Omega_i, \Sigma_i, \mu_i) .$$

We denote the measure space $(\Omega, \Sigma, \varepsilon \cdot \mu)$ by $\varepsilon \cdot (\Omega, \Sigma, \mu)$.

As a consequence of the Maharam theorem we obtain the following result, which is also called the Maharam theorem.

Theorem 1.14 (Maharam [87] (1942)). *For arbitrary finite atomless measure space (Ω, Σ, μ) there exists a unique, at most, countable collection of ordinals $\mathcal{M}$ and a decomposition $\Omega = \bigsqcup_{\alpha\in\mathcal{M}} \Omega_\alpha$ with $\Omega_\alpha \in \Sigma^+$, which is also unique, up to measure zero sets, such that $(\Omega_\alpha, \Sigma(\Omega_\alpha), \mu|_{\Sigma(\Omega_\alpha)})$ is isomorphic to $\varepsilon_\alpha \cdot D^{\omega_\alpha}$, where $\varepsilon_\alpha = \mu(\Omega_\alpha)$. In other words, (Ω, Σ, μ) is isomorphic to the direct sum $\sum_{\alpha\in\mathcal{M}} \oplus\, \varepsilon_\alpha \cdot (D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$.*

Observe that the isomorphism of the corresponding Boolean algebras $\widetilde{\Sigma}(\Omega_\alpha)$ and $\widetilde{\Sigma}_{\omega_\alpha}$ is induced by a map from Ω_α to D^{ω_α} in the proof of Maharam's theorem.

Definition 1.15. The collection of ordinals $\mathcal{M}$ that appears in Theorem 1.14 will be called the *Maharam set* of the measure space (Ω, Σ, μ).

A very important special case of the Maharam theorem is the Carathéodory theorem (see [75, p. 127]):

Theorem 1.16 (Carathéodory). *Every two separable atomless probability spaces are isomorphic.*

As a consequence, an r.i. space which is defined on $[0,1]$ can be considered, isometrically, as defined on $[0,1]^2$ or ω^α, whichever is more convenient.

1.5 Necessary information on vector lattices

For more information on vector lattices we refer the reader to [6] and [74]. A vector space (over the reals) E with a partial order $\le$ is called an *ordered vector space* if

- for all $x, y \in E$ if $x \le y$ then $x + z \le y + z$ for every $z \in E$;
- for all $x, y \in E$, if $x \le y$ then $\alpha x \le \alpha y$ for every $\alpha \ge 0$.

An element x of an ordered vector space E is called *positive* if $x \ge 0$, and the set of all positive elements of E is denoted by $E^+ = \{x \in E : x \ge 0\}$. A subset X of a partially ordered set E is called *order bounded* if there exists $M \in E^+$ such that $|x| \le M$ for each $x \in X$.

An element x_0 of a subset A of a partially ordered set E is called the *least upper bound* of A (and is denoted by $x_0 = \sup A$) if

- $x \le x_0$ for every $x \in A$;
- for every $y \in E$, if $x \le y$ for all $x \in A$ then $x_0 \le y$.

Analogously (by replacing the order direction $\le$ in the inequalities with $\ge$), we define the *greatest lower bound* of a subset A of a partially ordered set E, which is denoted by $\inf A$. An ordered vector space E is called a *vector lattice* (or, *Riesz space*) if for any two elements $x, y \in E$ there exists the least upper bound $x \vee y = \sup\{x, y\}$ and the greatest lower bound $x \wedge y = \inf\{x, y\}$ of the two-point set $\{x, y\}$ in E. Due to the vector structure, one may require the existence of the least upper bound of any two-point set in E in this definition. A vector lattice is called *Dedekind complete* if each bounded from above set has the least upper bound. A vector lattice E is called *Archimedean* if $\inf\{n^{-1}x : n \in \mathbb{N}\} = 0$ for each $x \in E^+$. We will only consider Archimedean vector lattices, which include all Dedekind complete vector lattices [129, p. 64]. A subset F of a vector lattice E is called *order closed*, if for any subset $G \subseteq F$ the existence of $y = \sup G \in E$ (or $y = \inf G \in E$) implies that $y \in F$.

Let E be a vector lattice. For every $x \in E$ the elements $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$ are called the *positive part*, the *negative part*, and the *modulus* of x, respectively. Observe that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Two elements $x, y \in E$ are called disjoint (or orthogonal) if $|x| \wedge |y| = 0$ and this fact is written as

$x \perp y$. It is not difficult to see that $x^+ \perp x^-$ for each $x \in E$. In general, $x = x^+ - x^-$ is not a unique representation of x as a difference of two positive elements. However, it is a unique representation of x as a difference of two disjoint positive elements. Two subsets $A, B \subseteq E$ are disjoint if $x \perp y$ for all $x \in A$ and $y \in B$. For any subset $A \subseteq E$ by A^d we denote the set $A^d = \{x \in E : A \text{ and } \{x\} \text{ are disjoint}\}$. The notation $x = \bigsqcup_{k=1}^n x_k$ means that $x = \sum_{k=1}^n x_k$ and $x_i \perp x_j$ if $i \neq j$.

Following [2, p. 86] we say that an element $u > 0$ of a vector lattice E is an *atom*, whenever $0 \le x \le u$, $0 \le y \le u$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$. For a Dedekind complete vector lattice E we can equivalently reformulate this definition as follows. A nonzero element x of a Dedekind complete vector lattice E is an *atom* if for each $y \in E$ the equality $|x| = |y|$ is possible only if $y = x$ or $y = -x$. A vector lattice E is *atomless* if there is no atom $x \in E$.

An element y of a vector lattice E is called a *fragment* (in other terminology, a *component*) of an element $x \in E$, provided $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. Evidently, a nonzero element $x \in E$ is an atom if and only if the only fragments of x are 0 and x itself. Hence, a Dedekind complete vector lattice E is atomless if each nonzero element $x \in E$ has a proper fragment $y \sqsubseteq x$, that is, $0 \neq y \neq x$. Clearly, any Köthe F-space on an atomless measure space is an atomless vector lattice.

A disjoint tree on a positive element of a vector lattice. The following object will be used in different constructions below. Let E be a vector lattice and $0 \neq e \in E^+$. A sequence $(e_n)_{n=1}^\infty$ in E is called a *disjoint tree* on e if $e_1 = e$ and $0 < e_n = e_{2n} \sqcup e_{2n+1}$ for each $n \in \mathbb{N}$. Clearly, all e_n are fragments of e.

The order convergence

We shall consider the *order convergence* in vector lattices. Here we give a brief introduction; for more details we refer the reader to [5].

Let E be a vector lattice. By a *net* in E we mean any function $\phi : \Lambda \to E$ from a *directed* partially ordered set Λ (that is, for each $\alpha, \beta \in \Lambda$ there is $\gamma \in \Lambda$ such that $\alpha \le \gamma$ and $\beta \le \gamma$) to E. Such a net is denoted by $(x_\alpha)_{\alpha \in \Lambda}$, where $x_\alpha = \phi(\alpha)$, or just by (x_α) if we are not interested in paying attention to a concrete set of indices Λ. A net $(x_\alpha)_{\alpha \in \Lambda}$ is called *increasing* (resp., *decreasing*) provided $x_\alpha \le x_\beta$ (resp., $x_\alpha \ge x_\beta$) for all indices $\alpha < \beta$ from Λ. In this case we write $x_\alpha \uparrow$ (resp., $x_\alpha \downarrow$).

A decreasing net (x_α) *order converges to zero* in E (notation $x_\alpha \downarrow 0$) if $\inf_\alpha x_\alpha = 0$. More generally, a net $(x_\alpha)_{\alpha \in \Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \le u_\beta$ for all $\beta \in \Lambda$. A map $f : E \to F$ between vector lattices is called *order continuous at a point* $x_0 \in E$ if for any net (x_α) in E the condition $x_\alpha \xrightarrow{\text{o}} x$ implies that $f(x_\alpha) \xrightarrow{\text{o}} f(x)$ in F. A map $f : E \to F$ is called *order continuous* if it is order continuous at each point $x_0 \in E$. We remark that the lattice operations of two variables $x + y$, $x - y$, ax (where $a \in \mathbb{R}$), $x \vee y$ and $x \wedge y$ and of one variables $x \to x^+$,

$x \to x^-$ and $x \to |x|$ are order continuous (to see that, one first needs to prove the inequalities $|x+y| \le |x|+|y|$, $||x|-|y|| \le |x-y|$, $|(x \vee z)-(y \vee z)| \le |x-y|$, and the same with $\vee$ and $\wedge$ for any $x, y, z \in E$). In particular, the order continuity of the first of these operations implies that, a linear map between vector lattices is order continuous if and only if it is order continuous at the origin. For the order convergence, one can also use the Squeeze theorem, and pass to a limit in an inequality for order convergent nets.

There is another way, in more natural terms for analysts, to introduce the notion of order convergence. The *lower* and the *upper order limits* of an order bounded net (x_α) in a Dedekind complete vector lattice E are defined as follows

$$\liminf_{\alpha} x_\alpha = \sup_{\alpha} \inf_{\beta \ge \alpha} x_\beta = \bigwedge_{\alpha} \bigvee_{\beta \ge \alpha} x_\beta, \quad \limsup_{\alpha} x_\alpha = \inf_{\alpha} \sup_{\beta \ge \alpha} x_\beta = \bigvee_{\alpha} \bigwedge_{\beta \ge \alpha} x_\beta .$$

It is a useful exercise to show that $\liminf_\alpha x_\alpha \le \limsup_\alpha x_\alpha$ for any order bounded net (x_α). We remark that an order convergent net need not be order bounded (consider, for example, the net of real numbers $(x_n)_{n \in \mathbb{Z}}$ defined by $x_n = -n$ for $n = -1, -2, \ldots$ and $x_n = 0$ for $n = 0, 1, 2, \ldots$). However, it is not an essential restriction to consider only order bounded nets.

Proposition 1.17. *For an order bounded net (x_α) in a Dedekind complete vector lattice E the following conditions are equivalent:*

(a) (x_α) *is order convergent;*

(b) $\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha$.

Moreover, if these conditions hold, the order limit equals the lower and the upper limits.

One can find a proof of Proposition 1.17 in [5, p. 323].

We will use the following characterization of the order convergence in Köthe F-spaces.

Proposition 1.18. *Let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ). A sequence (x_n) in E order converges to an element $x \in E$ if and only if (x_n) is order bounded in E and $x_n \to x$ a.e. on Ω.*

Proof. It is not a difficult technical exercise to show that for a decreasing sequence $y_n \downarrow$ in E, the conditions $\inf_n y_n = 0$ and $y_n \to 0$ a.e. on Ω are equivalent. So, the condition $|x_n - x| \le y_n \downarrow 0$ implies the order boundedness of (x_n) and that $x_n \to x$ a.e. on Ω. Now let (x_n) be order bounded and $x_n \to x$ a.e. on Ω. We set $y_n(t) = \sup_{m \ge n} |x_m(t) - x(t)|$ for each $t \in \Omega$. Then $|x_n - x| \le y_n$ and $y_n \downarrow$. By the above, $y_n \downarrow 0$. □

Definition 1.19. Let E be a vector lattice. For an arbitrary set J a series $\sum_{j\in J} x_j$ of elements $x_j \in E$ is called *order convergent* and the family $(x_j)_{j\in J}$ is called *order summable* if the net $(y_s)_{s\in J^{<\omega}}$, $y_s = \sum_{j\in s} x_j$ order converges to some $y_0 \in E$, where $J^{<\omega}$ is the net of all finite subsets $s \subseteq J$ ordered by inclusion. In this case y_0 is called the *order sum of the series* $\sum_{j\in J} x_j$ and we write $y_0 = \sum_{j\in J} x_j$. A series $\sum_{j\in J} x_j$ is called *absolutely order convergent* and the family $(x_j)_{j\in J}$ is called *absolutely order summable* if the series $\sum_{j\in J} |x_j|$ order converges.

We shall use some elementary properties of the above notions.

Proposition 1.20 ([92]). *Let E be a Dedekind complete vector lattice, $x_j \in E^+$, $j \in J$. Then we have the following:*

(i) *The order convergence of $\sum_{j\in J} x_j$ is equivalent to the order boundedness of $\{\sum_{j\in t} x_j : t \in J^{<\omega}\}$; in this case $\sum_{j\in J} x_j = \sup_{t\in J^{<\omega}} \sum_{j\in t} x_j$.*

(ii) *If the series $\sum_{j\in J} x_j$ order converges and $y_j \in E$ are such that $|y_j| \leq x_j$, for $j \in J$, then the series $\sum_{j\in J} y_j$ is also order convergent and $|\sum_{j\in J} y_j| \leq \sum_{j\in J} x_j$.*

Proof. To prove (i), it suffices to observe that, by positivity of the elements, for each $s \in J^{<\omega}$ we have $\sup_{t\geq s} \sum_{j\in t} x_j = \sup_{t\in J^{<\omega}} \sum_{j\in t} x_j$ if both corresponding sets are bounded. Moreover, the boundedness of these sets are equivalent.

(ii) Assume first that $y_j \geq 0$ for every $j \in J$. Then the set $\{\sum_{j\in t} y_j : t \in J^{<\omega}\}$ is bounded by $\sum_{j\in J} x_j$. Since E is Dedekind complete, there exist the least upper bound $\sum_{j\in J} y_j = \sup_{t\in J^{<\omega}} s \sum_{j\in t} y_j \leq \sup_{t\in J^{<\omega}} \sum_{j\in t} x_j = \sum_{j\in J} x_j$.

Now we consider the general case $|y_j| \leq x_j$. Let $y^+ = \sum_{j\in J} y_j^+$, $y^- = \sum_{j\in J} y_j^-$, $y = y^+ - y^-$. Then for each $s_0 \in J^{<\omega}$ we obtain

$$\begin{aligned}\inf_{s\geq s_0}\sup_{t\geq s}\sum_{j\in t} y_j &= \inf_{s\geq s_0}\sup_{t\geq s}\Big(\sum_{j\in t} y_j^+ - \sum_{j\in t} y_j^-\Big) \leq \inf_{s\geq s_0}\sup_{t\geq s}\Big(y^+ - \sum_{j\in s} y_j^-\Big)\\ &= \inf_{s\geq s_0}\Big(y^+ - \sum_{j\in s} y_j^-\Big) = y^+ - y^- = y.\end{aligned}$$

Analogously,

$$\begin{aligned}\sup_{s\geq s_0}\inf_{t\geq s}\sum_{j\in t} y_j &= \sup_{s\geq s_0}\inf_{t\geq s}\Big(\sum_{j\in t} y_j^+ - \sum_{j\in t} y_j^-\Big) \geq \sup_{s\geq s_0}\inf_{t\geq s}\Big(\sum_{j\in s} y_j^+ - y^-\Big)\\ &= \sup_{s\geq s_0}\Big(\sum_{j\in s} y_j^+ - y^-\Big) = y^+ - y^- = y.\end{aligned}$$

Thus, we have proved that $\sup_{s\geq s_0}\inf_{t\geq s}\sum_{j\in t} y_j \geq y \geq \inf_{s\geq s_0}\sup_{t\geq s}\sum_{j\in t} y_j$. On the other hand ,

$$\inf_{s\geq s_0}\sup_{t\geq s}\sum_{j\in t} y_j \geq \inf_{s\geq s_0}\sup_{t\geq s}\Big(\sum_{j\in t} y_j^+ - y^-\Big) = \inf_{s\geq s_0}(y^+ - y^-) = y\ .$$

Hence, $\inf_{s \geq s_0} \sup_{t \geq s} \sum_{j \in t} y_j = y$. Analogously, $\sup_{s \geq s_0} \inf_{t \geq s} \sum_{j \in t} y_j = y$. Finally, using (i), we obtain that

$$\Big|\sum_{j \in J} y_j\Big| = |y| = |y^+ - y^-| \leq y^+ + y^- = \sum_{j \in J} y_j^+ + \sum_{j \in J} y_j^-$$
$$= \sum_{j \in J} (y_j^+ + y_j^-) = \sum_{j \in J} |y_j| \leq \sum_{j \in J} x_j \,.$$

□

Corollary 1.21. *An absolutely order convergent series $\sum_{j \in J} x_j$ in a Dedekind complete vector lattice is order convergent and $|\sum_{j \in J} x_j| \leq \sum_{j \in J} |x_j|$.*

We shall show later that the absolute order convergence of a series of operators in L_1 is equivalent to the strong ℓ_1-sum of operators introduced by Rosenthal in [128]. But first we need to introduce the lattice structure in the operators space.

Lattices of linear operators and the modulus of an operator

Given vector lattices E and F, by $L(E, F)$ we denote the vector space of all linear operators $T : E \to F$. An operator $T \in L(E, F)$ is called:

- *positive* (we write $T \geq 0$) if $Tx \in F^+$ for every $x \in E^+$;
- *regular* if T is a difference of two positive operators $T = S_1 - S_2$, $S_1, S_2 \geq 0$;
- *order bounded* if T sends order bounded sets in E to order bounded sets in F.

The symbols $L^+(E, F)$, $L_r(E, F)$ and $L_b(E, F)$ will be used to denote the sets of all positive, regular and order bounded linear operators from an ordered vector space E to an ordered vector space F. Since every positive operator is obviously order bounded, we have that $L_r(E, F) \subseteq L_b(E, F)$ for arbitrary ordered vector spaces E and F.

We define an order on $L(E, F)$ by setting $S \leq T$ if and only if $T - S \geq 0$. It is an easy exercise to show that $L(E, F)$ is an ordered vector space with respect to the defined above order. However, this ordered vector space need not be a vector lattice, in general. The problem is, that $S \vee T$ and $S \wedge T$ may not exist. It is interesting to notice that the lattice operations $\vee$ and $\wedge$ can be described by the modulus of an element. More precisely,

(i) $x \vee y = \frac{1}{2}(x + y + |x - y|)$

(ii) $x \wedge y = \frac{1}{2}(x + y - |x - y|)$

hold for each $x, y \in E$. Indeed,

$$\begin{aligned} x + y + |x - y| &= x + y + (x - y) \vee (y - x) \\ &= \big((x + y) + (x - y)\big) \vee \big((x + y) + (y - x)\big) \\ &= (2x) \vee (2y) = 2(x \vee y), \end{aligned}$$

and (ii) is proved analogously. Thus, if for every element x of an ordered vector space E there exists its modulus $|x| = x \vee (-x)$ then E is a vector lattice.

An answer to the question of when does the ordered vector space of all order bounded linear operators is a vector lattice, is given by the following theorem, see [6, p. 12].

Theorem 1.22 (Riesz–Kantorovich's theorem). *Let E, F be vector lattices with F Dedekind complete. Then the ordered vector space $L_b(E, F)$ is a Dedekind complete vector lattice. Its lattice operations satisfy the following equalities for every $S, T \in L_b(E, F)$ and $x \in E^+$*

$$\big(S \vee T\big)\, x = \sup\big\{Sy + Tz : y, z \in E^+,\ x = y + z\big\};$$
$$\big(S \wedge T\big)\, x = \inf\big\{Sy + Tz : y, z \in E^+,\ x = y + z\big\}.$$

Thus, if E, F are vector lattices with F Dedekind complete, then for any operator $T \in L(X, Y)$ the following assertions are equivalent:

(a) T is regular.

(b) T is order bounded (in particular, $|T|$ exists by Dedekind completeness of F).

Moreover, if these conditions are satisfied then $|T|x = \sup\{|Ty| : |y| \le x\}$ for each $x \in E^+$.

The following statement gives useful formulas for evaluating the modulus of an operator (see [6, p. 15] and [6, p. 39]).

Lemma 1.23. *Let E and F be vector lattices with F Dedekind complete. Then for every $T \in L_b(E, F)$ and every $x \in E^+$ we have*

$$|T|\, x = \sup\Big\{\sum_{k=1}^{n} \big|T x_i\big| : x = \sum_{k=1}^{n} x_k,\ \ x_k \in E^+,\ \ n \in \mathbb{N}\Big\}. \qquad (1.3)$$

If, moreover, E is Dedekind complete then

$$|T|\, x = \sup\Big\{\sum_{k=1}^{n} \big|T x_i\big| : x = \bigsqcup_{k=1}^{n} x_k,\ \ x_k \in E^+,\ \ n \in \mathbb{N}\Big\}. \qquad (1.4)$$

Furthermore, the expression in the braces of the right-hand side of both equalities is an increasing net with respect to the directed set of all finite partitions of x into disjoint fragments, ordered as follows: $(x_i)_{i \in I} \le (y_j)_{j \in J}$ if and only if $x_i = \bigsqcup_{j \in J_i} y_j$ for each $i \in I$ and some subset $J_i \subseteq J$.

Ideals and bands

A *sublattice* of a vector lattice E is a vector subspace F of E which is a lattice itself with respect to the same ordering. A subset A of a vector lattice E is called *solid* if for any $x \in A$ and $y \in E$ the condition $|y| \le |x|$ implies that $y \in A$. A solid vector subspace is called an *ideal*. An order closed ideal is called a *band*. A band I of a vector lattice E is called a *projection band* if $E = I \oplus I^d$.

For an arbitrary vector lattice E and any subset $A \subseteq E$ by Band(A) we denote the least band in E which contains A (obviously, the intersection of bands is a band, so, Band(A) equals the intersection of all bands of E containing A). The following useful result is well known, see [6, p. 33], [129, p. 62].

Lemma 1.24. *Let E be a Dedekind complete vector lattice, $A \subseteq E$ be any subset. Then A^d is a band and $E = \mathrm{Band}(A) \oplus A^d$. In particular, each band is a projection band.*

Let A be a solid subset of a vector lattice E. We denote by Abs(A) the set of all sums of absolutely order convergent series $\sum_{j\in J} x_j$ of elements $x_j \in A$.

We will frequently use the following result. It should be noted that the idea of the statement was implicitly used by Rosenthal in [128] for the case of the lattice $\mathcal{L}(L_1)$ to obtain a decomposition of any operator $T \in \mathcal{L}(L_1)$ as a sum $T = T_a + T_c$ of a purely atomic and a purely continuous operators, however for the setting of vector lattices it appeared in papers of O. Maslyuchenko, Mykhaylyuk and Popov [92, 93].

Theorem 1.25. *Let A be a solid subset of a Dedekind complete vector lattice E. Then*

(i) $A^d = \{x \in E : \textit{for all } y \in A, \quad 0 \le y \le |x| \textit{ implies } y = 0\}$;

(ii) $\mathrm{Band}(A) = \mathrm{Abs}(A)$.

In particular, $E = \mathrm{Abs}(A) \oplus A^d$ is a decomposition into mutually complemented bands.

Proof. Let B denote the set in the right-hand-side of (i).

(i) Suppose that $x \in A^d$, $y \in A$ and $0 \le y \le |x|$. Then $0 = |x| \wedge y = y$ and hence $x \in B$.

Now suppose that $x \in B$ and $y \in A$. Since A is solid and $0 \le |x| \wedge |y| \le |y|$, we have that $|x| \wedge |y| \in A$. By the definition of B and $0 \le |x| \wedge |y| \le |x|$ we have that $|x| \wedge |y| = 0$ and hence $x \in A^d$.

(ii) By Lemma 1.24,

$$E = \mathrm{Band}\,(A) \oplus A^d\ . \tag{1.5}$$

We claim that

$$E = \mathrm{Abs}\,(A) \oplus A^d\ . \tag{1.6}$$

First we prove the existence of an expansion. Let $x \in E$. Since $x = x^+ - x^-$, it is enough to consider the case $x > 0$. Observe that for each summable family $\sum_{i\in I} x_i$, $x_i \in E$, and each $a \in E$, $a \ne 0$, the set $I_a = \{i \in I : x_i = a\}$ is finite. Let S be a set of cardinality card$(S) >$ card(E). Consider the collection

$$\mathcal{A}_x = \Big\{(x_j)_{j\in J} : J \subseteq S,\ x_j \in A,\ x_j > 0,\ \sum_{j\in J} x_j \le x\Big\}\,,$$

with the order $(x_i)_{i\in I} \le (y_j)_{j\in J}$ if $I \subseteq J$ and $x_i = y_i$ for $i \in I$. Obviously, $I \ne S$ for each family $(x_i)_{i\in I} \in \mathcal{A}_x$. If $\mathcal{A}_x = \emptyset$ then it is easy to see that $x \in B$. Therefore, (i) implies that $x \in A^d$ and $x = 0 + x$ is the desired expansion. Let now $\mathcal{A}_x \ne \emptyset$. Since the inequality $\sum_{j\in J} x_j \le x$ is equivalent to the inequalities $\sum_{j\in J_0} x_j \le x$ for any finite subset $J_0 \subseteq J$ then the family $(\mathcal{A}_x, \le)$ is inductively ordered. By the Zorn lemma, there exists a maximal element $(a_j)_{j\in J}$ in $\mathcal{A}_x$. Then $x_1 = \sum_{j\in J} a_j \in \mathrm{Abs}\,(A)$ and $x_1 \le x$. Using (i) and the maximality of $(a_j)_{j\in J}$ we obtain that $x_2 = x - x_1 \in A^d$. Thus, $x = x_1 + x_2$ is the desired expansion. Since $\mathrm{Abs}\,(A) \subseteq \mathrm{Band}\,(A)$, (1.5) implies that $\mathrm{Abs}\,(A) \cap A^d = \{0\}$. Thus, (1.6) is proved. It remains to note that $\mathrm{Abs}\,(A) \subseteq \mathrm{Band}\,(A)$, (1.5) and (1.6) together imply that $\mathrm{Abs}\,(A) = \mathrm{Band}\,(A)$. □

We remark that since $\mathrm{Abs}\,(A)$ is a band, it is equal to the set of all sums of (not necessary absolutely) order convergent series $\sum_{j\in J} x_j$ of elements $x_j \in A$.

Banach lattices

A Banach space X which is a vector lattice with respect to an order $\le$ is called a *Banach lattice* if for each $x, y \in X$ the inequality $|x| \le |y|$ implies that $\|x\| \le \|y\|$.

By definition, every Köthe–Banach space is a Banach lattice with respect to the order $x \le y$ whenever $x(t) \le y(t)$ a.e. Conversely, every Banach lattice, under minor assumptions, in certain sense can be considered as a Köthe–Banach space. To make it precise, we recall some definitions. Let F be an ideal of a Banach lattice E. An element $e \in F^+$ is called a *weak unit* of F provided for every $x \in F$ the condition $x \perp e = 0$ implies that $x = 0$. Two Banach lattices are called *order isometric* if there is an order-preserving linear isometry between them, called an *order isometry*. A Banach lattice E is called *order continuous* if for each net (x_α) in E the condition $x_\alpha \downarrow 0$ implies that $\|x_\alpha\| \to 0$. Note that in this case the condition $x_\alpha \xrightarrow{\text{o}} 0$ also implies that $\|x_\alpha\| \to 0$. If the same is true for sequences only, E is said to be *σ-order continuous*.

Proposition 1.26. *Let E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ). Then the Banach lattice E is σ-order continuous if and only if E has an absolutely continuous norm.*

Proof. Assume E is σ-order continuous. Then, given any $x \in E$, $A_n \in \Sigma$, $A_{n+1} \subseteq A_n$, $\bigcap_{n=1}^\infty A_n = \emptyset$, then $|x| \cdot \mathbf{1}_{A_n} \downarrow 0$ and hence, $\|x \cdot \mathbf{1}_{A_n}\| \to 0$, that is, E has an absolutely continuous norm.

Let E have an absolutely continuous norm, $x_n \in E$, $x_n \downarrow 0$. We prove that $\|x_n\| \to 0$. Assuming the contrary, we choose a subsequence (y_n) of (x_n) so that $\|y_n\| \ge \delta$ for all n and some $\delta > 0$. Since $x_n \downarrow 0$, by Proposition 1.18, $y_n \to 0$ a.e. on Ω, hence, there is a subsequence (z_n) of (y_n) such that $z_n \xrightarrow{\mu} 0$. In particular, $\mu(A_n) \to 0$ where $A_n = \{t \in \Omega : |z_n(t)| \ge \delta/2\}$. By the absolute continuity of norm, $\|x_1 \cdot \mathbf{1}_{A_n}\| \to 0$. Let n_0 be such that $\|x_1 \cdot \mathbf{1}_{A_{n_0}}\| < \delta/2$. Since $|z_{n_0}| \le |x_1|$, we have that $\|z_{n_0} \cdot \mathbf{1}_{A_{n_0}}\| \le \|x_1 \cdot \mathbf{1}_{A_{n_0}}\| < \delta/2$ and hence ,

$$\left\|z_{n_0}\right\| \le \left\|z_{n_0} \cdot \mathbf{1}_{A_{n_0}}\right\| + \left\|z_{n_0} \cdot \mathbf{1}_{\Omega \setminus A_{n_0}}\right\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \,,$$

a contradiction. □

Below we mention two results which imply that the notion of a Banach lattice is not far from Köthe–Banach spaces.

Theorem 1.27. (Kakutani, [80, Proposition 1.a.9]) *Every order continuous Banach lattice X can be decomposed into an unconditional direct sum of a (generally uncountable) family of disjoint ideals $(X_\lambda)_{\lambda\in\Lambda}$, each X_λ having a weak unit $e_\alpha > 0$. More precisely, every $x \in X$ has a unique representation $x = \sum_{\lambda\in\Lambda} x_\lambda$, with $x_\lambda \in X_\lambda$ for all $\lambda \in \Lambda$, where the series has, at most, countably many nonzero terms and converges unconditionally. Moreover, if Z is a separable subspace of X then one of the indices, say λ_0, can be chosen so that $Z \subseteq X_{\lambda_0}$.*

Corollary 1.28. *Every separable subspace Z of a Banach lattice X is contained in some ideal X_0 of X with a weak unit.*

Theorem 1.29. ([80, Theorem 1.b.14]) *Let X be an order continuous (= σ-Dedekind complete and σ-order continuous) and Banach lattice with a weak unit. Then there exists a probability space (Ω', Σ', μ'), an ideal (not necessarily closed) Y of $L_1(\mu')$ and a lattice norm $\|\cdot\|_Y$ on Y such that*

(a) *there is an order isometry $J : (Y, \|\cdot\|_Y) \to X$;*

(b) *$L_\infty(\mu')$ is dense in Y, and Y is dense in $L_1(\mu')$;*

(c) *$\|x\|_1 \le \|x\|_Y \le 2\|x\|_\infty$ for all $x \in L_\infty(\mu')$;*

(d) *The dual Y^* to $(Y, \|\cdot\|_Y)$ has the following representation: for every $G \in Y^*$ there exists a unique element $g \in Y'$ such that $G(y) = \int_{\Omega'} yg \,\mathrm{d}\mu'$ for every $y \in Y$, and moreover, $\|G\| = \|g\|$.*

Combined Theorems 1.27 and 1.29, give a representation of an order continuous Banach lattice as a Köthe–Banach space.

The following simple observation is very useful.

Proposition 1.30. *If Y and Z are mutually complemented bands in a Banach lattice X then the decomposition $X = Y \oplus Z$ generates contractive projections, that is, if $x = y + z$ with $y \in Y$ and $z \in Z$ then $\max\{\|y\|, \|z\|\} \le \|x\|$.*

Indeed, it is a useful exercise to show that if $u \perp v$ in a vector lattice then $|u + v| = |u| + |v|$. In particular, $|x| = |y| + |z|$ and hence, $|y| \le |x|$ and $|z| \le |x|$. By the definition of a Banach lattice, $\|y\| \le \|x\|$ and $\|z\| \le \|x\|$.

1.6 Kalton's and Rosenthal's representation theorems for operators on L_1 and their generalization to vector lattices

It is a very special and very important property of operators on L_1 that every operator $T \in \mathcal{L}(L_1)$ is regular [129, p. 232], that is, it is equal to a difference of two positive operators. As a consequence, we obtain that for any operator $T \in \mathcal{L}(L_1)$ the modulus $|T| \in \mathcal{L}(L_1)$ exists and could be defined by (1.3) (or equivalently, by (1.4)). Moreover, $\||T|\| = \|T\|$ for every $T \in \mathcal{L}(L_1)$ [129, p. 232]. This latter fact yields that the vector lattice $\mathcal{L}(L_1)$ is a Banach lattice with respect to the operator norm.

The following representation of an operator on L_1 as a sum of its "large" and "small" parts was obtained by Kalton in [66] (1978).

Theorem 1.31 (Kalton's representation theorem). *For any operator $T \in \mathcal{L}(L_1)$ there exists a weak*-measurable function μ_t from $[0,1]$ to the set $M[0,1]$ of all regular Borel measures on $[0,1]$, such that for each $x \in L_1$*

$$Tx(t) = \int_{[0,1]} x(\tau)\,\mathrm{d}\mu_t(\tau) \quad a.e. \tag{1.7}$$

Conversely, every weak-measurable function $\mu_t : [0,1] \to M[0,1]$ defines an operator $T \in \mathcal{L}(L_1)$ by (1.7).*

Once we decompose the measure μ_t which defines an operator $T \in \mathcal{L}(L_1)$ as a sum of the atomic part $\mu_t^a = \sum_{n=1}^{\infty} a_n(t)\delta_{\sigma_n(t)}$ (where δ_τ is Dirac's measure) and the atomless part μ_t^c, we get the following representation of T

$$Tx(t) = \sum_{n=1}^{\infty} a_n(t)x(\sigma_n(t)) + \int_{[0,1]} x(\tau)\,\mathrm{d}\nu_t(\tau)\,, \tag{1.8}$$

(see also [45]).

In the proof of the main result of [128] Rosenthal used a theorem which we call Rosenthal's version of the Kalton representation theorem. For some purposes it appears to be more convenient than the original Kalton's theorem. As noted by Rosenthal in [128], his version could be formally obtained from Kalton's theorem. This theorem of Rosenthal can be stated as follows. Each operator $T \in \mathcal{L}(L_1)$ can be uniquely represented as $T = T_{pa} + T_c$ where $T_{pa}, T_c \in \mathcal{L}(L_1)$ are purely atomic and purely continuous operators, respectively (for the definitions see below).

We present below a generalized version of Rosenthal's representation theorem. This version (Theorem 1.33) is a partial case of Theorem 1.25 applied to the set $A = L_{dpo}(E,F)$ of disjointness-preserving operators acting from a vector lattice E to a Dedekind complete vector lattice F, as a solid subspace of the lattice $L_r(E,F)$ of all regular linear operators from E to F. Thus, we have a decomposition of $L_r(E,F)$

into mutually complemented bands $L_r(E, F) = L_{pe} \oplus L_{pn}$, where L_{pe} is the band generated by disjointness-preserving operators (notation L_{pe} will be explained in Definition 1.32 below), and L_{pn} is the band of the so-called *pseudonarrow operators* (see Definition 1.32 below). It is an amazing and deep result that the band L_{pn} coincides with the set of all regular narrow operators from E to F (for details see Chapter 10). This last result for $E = F = L_1$ was established by Kalton and Rosenthal, using another terminology.

A generalization of Rosenthal's theorem to regular operators on any lattice

Recall that an operator $T \in L(E, F)$ is called *disjointness preserving* (d.p.o., in short) if T maps disjoint elements to disjoint elements. The set of all disjointness-preserving regular operators is denoted by $L_{dpo}(E, F)$.

Definition 1.32. Let E, F be vector lattices with F Dedekind complete. An operator $T \in L_r(E, F)$ is called

- a *pseudo-embedding* if there exists an absolutely order summable family $(T_j)_{j \in J}$ of d.p.o. in $L_r(E, F)$ such that $T = \sum_{j \in J} T_j$;
- *pseudonarrow* if there is no nonzero d.p.o. $S \in L_r^+(E, F)$ with $S \leq |T|$.

A linear operator between two vector lattices is a positive d.p.o. if and only if it is a *lattice homomorphism*, that is, a linear operator which preserves the lattice operations (see [6, p. 88]).

Our terminology "pseudo-embedding" and "pseudonarrow operator" will become clear in Chapter 7 due to two theorems of Rosenthal concerning operators on L_1. One of them (Theorem 7.39) asserts that a nonzero operator is a pseudo-embedding if and only if it is a near isometric embedding when restricted to a suitable $L_1(A)$-subspace. The other one (Theorem 7.45) implies, in particular, that an operator in L_1 is narrow if and only if it is pseudonarrow.

The set of all pseudo-embeddings from E to F will be denoted $L_{pe}(E, F)$. Thus, $L_{pe}(E, F) = \mathrm{Abs}(L_{dpo}(E, F))$ by definition. The set of all pseudonarrow operators $T \in L_r(E, F)$ will be denoted $L_{pn}(E, F)$.

Theorem 1.33. *Let E, F be vector lattices with F Dedekind complete. Then*

(i) *$L_{dpo}(E, F)$ is solid in $L_r(E, F)$;*

(ii) *$\mathrm{Band}\,(L_{dpo}(E, F)) = L_{pe}(E, F)$;*

(iii) *$L_{dpo}(E, F)^d = L_{pn}(E, F)$;*

(iv) *$L_{pe}(E, F)$ and $L_{pn}(E, F)$ are mutually complemented bands, hence $L_r(E, F) = L_{pe}(E, F) \oplus L_{pn}(E, F)$.*

Proof. By Theorem 1.25, it is enough to prove (i). Suppose $0 \le A \in L_{dpo}(E,F)$, $B \in L_r(E,F)$ and $|B| \le A$. We prove that $B \in L_{dpo}(E,F)$. Let $x_1, x_2 \in E$, $x \perp y$. Since $|Bx_i| \le |B|\,|x_i| \le A\,|x_i|$, $i = 1, 2$, we obtain that $0 \le |Bx_1| \wedge |Bx_2| \le A\,|x_1| \wedge A\,|x_2| = 0$. Thus, $Bx_1 \perp Bx_2$. □

Observe that $\operatorname{Band}(L_{dpo}(E,F)) = L_{pe}(E,F)$ is equal to the band generated by the lattice homomorphisms $T : E \to F$.

By the Ogasawara theorem [6, Theorem 4.4], the set $L_b^{oc}(E,F)$ of all order continuous order bounded operators from E to F is a band in $L_b(E,F) = L_r(E,F)$. Since an intersection of bands is a band, we obtain the following version of Theorem 1.33 for order continuous operators.

Corollary 1.34. *Let E, F be vector lattices with F Dedekind complete. Then*

(i) *set $L_{dpo}^{oc}(E,F)$ is solid in $L_r^{oc}(E,F)$;*

(ii) $\operatorname{Band}(L_{dpo}^{oc}(E,F)) = L_{pe}^{oc}(E,F)$*;*

(iii) $L_{dpo}^{oc}(E,F)^d = L_{pn}^{oc}(E,F)$*;*

(iv) *the sets $L_{pe}^{oc}(E,F)$ and $L_{pn}^{oc}(E,F)$ are mutually complemented bands, hence $L_r^{oc}(E,F) = L_{pe}^{oc}(E,F) \oplus L_{pn}^{oc}(E,F)$.*

Here, $L_{dpo}^{oc}(E,F)$, $L_{pe}^{oc}(E,F)$ and $L_{pn}^{oc}(E,F)$ denote the corresponding intersections of $L_{dpo}(E,F)$, $L_{pe}(E,F)$ and $L_{pn}(E,F)$ with $L_r^{oc}(E,F) = L_b^{oc}(E,F)$.

Chapter 2

Each "small" operator is narrow

In this chapter we identify classes of "small" operators which are contained in the class of narrow operators. The results are valid for operators on Köthe F-spaces with absolutely continuous norms. An absolute continuity assumption can be slightly relaxed (see Section 2.1) but it cannot be omitted: as we will see later, there are continuous linear functionals on L_∞ that are not narrow (see Section 10.2).

We also show that, on the one hand, narrow operators are very "small," since they allow a restriction which is a compact operator on a suitable subspace of the domain space. On the other hand, they are "large" in a sense that on some spaces (e.g. L_p, $1 < p < \infty$), a sum of two narrow operators can be an arbitrary operator (see Section 5.1).

Further analysis of what notions of "smallness" imply that the operator is narrow, is presented in Chapter 7.

2.1 AM-compact and Dunford–Pettis operators are narrow

A linear operator T from a vector lattice (in particular, from a Köthe function F-space) E to an F-space X is called *AM-compact* if T sends order bounded sets in E to relatively compact sets in X. If E is a Banach lattice then an AM-compact operator to a Banach space is automatically continuous. Obviously, each compact operator is AM-compact, but the converse is not true (for example, the conditional expectation operator in $L_p(\mu)$ for $1 \le p < \infty$ with respect to a purely atomic sub-σ-algebra).

Proposition 2.1. *Let E be a Köthe F-space with an absolutely continuous norm on the unit, and let X be an F-space. Then each compact and each AM-compact operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Given any $A \in \Sigma^+$ and $\varepsilon > 0$, we consider a Rademacher system (r_n) in $E(A)$. Then the set $\{Tr_n : n \in \mathbb{N}\}$ is relatively compact and hence, there are numbers $n \ne m$ such that $\|Th\| < \varepsilon$, where $h = (r_n - r_m)/2$. Since h is a sign on some $B \in \Sigma(A)$ with $\mu(B) = \mu(A)/2$, by Proposition 1.9, T is narrow. □

Let X and Y be separable Banach spaces. Using a standard technique, one can construct a compact injective operator $T \in \mathcal{L}(X, Y)$. Thus, we obtain the following consequence of Proposition 2.1.

Proposition 2.2. *Let E be a Köthe–Banach space with an absolutely continuous norm on the unit, and let X be a Banach space. Then there exists a narrow operator $T \in \mathcal{L}(E, X)$ which is not strictly narrow.*

An operator $T \in \mathcal{L}(X, Y)$ between Banach spaces X and Y is called a *Dunford–Pettis operator* if T sends weakly null sequences in X to norm null sequences in Y.

Proposition 2.3. *Let E be an r.i. Banach space on* $[0, 1]$ *which is not equal to L_∞, up to an equivalent norm, and let X be a Banach space. Then every Dunford–Pettis operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Given any $A \in \Sigma^+$, consider a Rademacher system (r_n) in $E(A)$. Since $r_n \overset{w}{\longrightarrow} 0$ [80, p. 160], we have that $\|Tr_n\| \to 0$. □

Let X be a Banach space. An operator $T \in \mathcal{L}(L_1(\mu), X)$ is said to be *representable* if there is $y \in L_\infty(X)$ such that $Tx = \int_\Omega xy \, d\mu$ for all $x \in L_1(\mu)$. Note that in this case $\|T\| = \|y\|$. For more information on representable operators we refer the reader to [29].

Proposition 2.4. *Let X be a Banach space. Then each representable and hence, each weakly compact operator $T \in \mathcal{L}(L_1(\mu), X)$ is narrow.*

Proof. Each representable operator is Dunford–Pettis [29, p. 74], and each weakly compact operator defined on $L_1(\mu)$ is representable [29, p. 75]. □

Let X and Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is called *absolutely summing* if T sends unconditionally convergent series in X to absolutely convergent series in Y.

Proposition 2.5. *Let E be a Köthe–Banach space and X be a Banach space. Suppose that for each $A \in \Sigma^+$ there exists a Rademacher system on A which is equivalent to the unit vector basis of ℓ_2. Then each absolutely summing operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Fix $A \in \Sigma^+$. Let (r_k) be a Rademacher system on A equivalent to the unit vector basis of ℓ_2. The unconditional convergence of the series $\sum_{n=1}^\infty \frac{1}{n} r_n$ implies that $\sum_{n=1}^\infty \frac{1}{n} \|Tr_n\| < \infty$ and hence $\liminf_n \|Tr_n\| = 0$. □

We remark that if E is an r.i. space on $[0, 1]$ with $q_E < \infty$ then the assumption of Proposition 2.5 is satisfied [80, p. 134].

Another well-known class of "small" operators is the class of strictly singular operators.

Let X, Y, Z be Banach spaces. We say that an operator $T \in \mathcal{L}(X, Y)$ *fixes a copy of* Z provided there exists a subspace X_0 of X isomorphic to Z such that the restriction $T|_{X_0}$ of T to X_0 is an isomorphic embedding. Otherwise we say that T

is Z-*strictly singular*. An operator $T \in \mathcal{L}(X, Y)$ is called *strictly singular* if it is Z-strictly singular for every Banach space Z.

Clearly, every compact operator is strictly singular, however the converse is not true: the identity operator from ℓ_p to ℓ_r with $1 \le p < r < \infty$ is strictly singular and noncompact. We will see later (Section 4.2) that narrow operators do not have to be strictly singular. However it is a very interesting problem whether every strictly singular operator has to be narrow.

Open problem 2.6. Let E be a Köthe–Banach space with an absolutely continuous norm, and X be a Banach space. Is every strictly singular operator $T \in \mathcal{L}(E, X)$ narrow?

This problem will be studied in detail in Chapter 7. In particular we will show a result of Bourgain and Rosenthal which says that the answer is affirmative for $E = L_1(\mu)$ (Theorem 7.2). There are also other partial results; however, in general the problem remains open. A related problem was posed by Plichko and Popov in [110].

Open problem 2.7. Let E be a Köthe–Banach space with an absolutely continuous norm, and X be a Banach space. Is every ℓ_2-strictly singular operator $T \in \mathcal{L}(E, X)$ narrow?

See Sections 9.5 and 10.9 for partial answers to this problem.

2.2 "Large" subspaces are exactly strictly rich

In the real case we have the following convenient criteria for a subspace to be strictly rich.

Theorem 2.8 ([110] (1990)). *Let E be a Köthe F-space over the reals on a finite atomless measure space (Ω, Σ, μ) for which there exists a reflexive Köthe–Banach space E_1 on (Ω, Σ, μ) with continuous inclusion embedding $E_1 \subseteq E$. Then for a subspace X of E the following assertions are equivalent:*

(i) $X \cap L_\infty(A) \neq \{0\}$ *for each* $A \in \Sigma^+$.

(ii) *X is strictly rich.*

(iii) *For each $A \in \Sigma^+$ and any number $\nu > 0$ there is (a "biased" sign) $x \in X$ of the form $x = \mathbf{1}_B - \nu \mathbf{1}_C$, where $A = B \sqcup C$ and $\mu(C) = \mu(A)/(\nu + 1)$.*

Notice that all spaces $E = L_p(\mu)$ with $0 \le p < \infty$ satisfy the assumption of the theorem, for example, with $E_1 = L_r(\mu)$ where $r = \max\{p, 2\}$. Moreover, this is also true for any r.i. Banach space with an absolutely continuous norm, because of a result of Semenov from [130] (1968).

For the proof of the theorem we need the following simple observation that will also be used later.

A subset $B = \{e_i : i \in I\}$ of a vector space E over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called a *Hamel basis* in E, if for every $x \in E$ there exists a unique collection of at most, finitely many nonzero scalars $(a_i)_{i \in I}$ such that $x = \sum_{i \in I} a_i e_i$. As a direct consequence of Zorn's lemma, we have that any vector space has a Hamel basis. One can prove that any two distinct Hamel bases of the space X are of the same (finite or infinite) cardinality κ, that is defined to be the *H-dimension* of X, i.e. $\text{H-dim}\, X \stackrel{\text{def}}{=} \kappa$ (see, e.g. [28, p. 3]). We use the notation H-dim X, because when X is a topological vector space, $\dim X$ is the usual notation for the least cardinality of subsets of X with dense linear span, that is not the same in the infinite dimensional case. The *H-codimension* of a subspace Z of a vector space X is defined to be the H-dimension of the quotient space $\text{H-codim}\, Z \stackrel{\text{def}}{=} \text{H-dim}\, X/Z$.

Lemma 2.9. *Let X be a vector space, and let Y, Z be its subspaces. If* $\dim Y >$ $\operatorname{codim} Z$ *then* $Y \cap Z \neq \{0\}$.

Proof of Lemma 2.9. Let $\tau : X \to X/Z$ be the quotient map and $\tau|_Y : Y \to X/Z$ its restriction to Y. Since $\dim Y > \dim X/Z$, there is $y \in Y \setminus \{0\}$ with $\tau y = 0$. This means that $y \in Y \cap Z \setminus \{0\}$. □

Proof of Theorem 2.8. Implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. We prove (i) $\Rightarrow$ (iii). For every $A \in \Sigma^+$ we set $E^0(A) = \{x \in E(A) : \int_\Omega x \, d\mu = 0\}$. Since μ is atomless, (i) yields that $X \cap L_\infty(A)$ is an infinite dimensional subspace of $E(A)$, and $E^0(A)$ is a 1-codimensional subspace of $E(A)$. Thus, by Lemma 2.9,

$$X \cap L_\infty^0(A) = X \cap L_\infty(A) \cap E^0(A) \neq \{0\} . \tag{2.1}$$

Fix any $\nu > 0$ and set $K_A = \{x \in X \cap L_\infty^0(A) : -\nu \le x \le 1\}$. By (2.1) $K_A \neq \{0\}$. Direct verifications show that K_A is a convex closed bounded subset of E. From the definition of a Köthe space we deduce that K_A is bounded in E_1, and by continuity of the embedding $E_1 \subseteq E$ we obtain that K_A is closed in E_1. By the Banach–Alaoglu theorem, K_A is a convex weakly compact subset of E_1, and by the Krein–Milman theorem, there exists an extreme point $x \in K_A$. We claim that, $x(\omega) \neq 0$ for almost all $\omega \in A$ and, moreover, if $x(\omega) > 0$ then $x(\omega) = 1$, and if $x(\omega) < 0$ then $x(\omega) = -\nu$ a.e. Indeed, otherwise there exist $\delta > 0$ and $B \in \Sigma^+(A)$ such that $-\nu(1-\delta) \le x(\omega) \le 1-\delta$ for every $\omega \in B$. Choose any $y \in K_B \setminus \{0\}$ and set $\delta' = \min\{\delta, \delta\nu, \delta/\nu\}$. Then $x + \delta' y \in K_A$ and $x - \delta' y \in K_A$. This contradicts the condition that x is an extreme point of K_A. Set $B = \{\omega \in A : x(\omega) = 1\}$ and $C = A \setminus B$. Then $x = \mathbf{1}_B - \nu \mathbf{1}_C$ and (iii) holds. □

As a consequence, we obtain that, under the theorem assumptions on E, every finite codimensional subspace F of E is strictly rich.

2.3 Operators with "small" ranges are narrow

Here by "small" we mean that the density of the range of an operator $T : E \to X$ is strictly less than the density of every subspace $E(A)$ with $A \in \Sigma^+$.

Before continuing to the main results, we establish a connection between absolute continuity of the norm and the denseness of simple functions.

Proposition 2.10 ([69]). *Let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ). If the norm of E is absolutely continuous then the set of all simple (= finite valued) functions is dense in E. Conversely, if the set of all simple functions is dense in E and E has an absolutely continuous norm on the unit then the norm of E is absolutely continuous.*

Proof. Throughout the proof, S denotes the linear space of all simple functions in E.

Let the F-norm on E be absolutely continuous. Observe that for each $x \in L_\infty(\mu)$, since $|x| \le \|x\|_{L_\infty(\mu)} \cdot \mathbf{1}_\Omega$ a.e., we have that $\|x\|_E \le \|\mathbf{1}_\Omega\| \|x\|_{L_\infty(\mu)}$. This implies that, since S is dense in $L_\infty(\mu)$ with respect to the norm of $L_\infty(\mu)$, S is dense in $L_\infty(\mu)$ with respect to the F-norm of E. Thus, it is enough to prove that $L_\infty(\mu)$ is dense in E. Fix any $x \in E$ and $\varepsilon > 0$. For each $n \in \mathbb{N}$ we set $\Omega_n = \{t \in \Omega : |x(t)| \le n\}$. By absolute continuity of the F-norm on E, there exists $n \in \mathbb{N}$ so that $\|x - x \cdot \mathbf{1}_{\Omega_n}\| = \|x \cdot \mathbf{1}_{\Omega \setminus \Omega_n}\| < \varepsilon$. Since $x \cdot \mathbf{1}_{\Omega_n} \in L_\infty(\mu)$, $L_\infty(\mu)$ is dense in E.

Now assume that the norm on E is an absolutely continuous norm on the unit, and S is dense in E. Fix any $x \in E$ and $\varepsilon > 0$. Choose $y \in S$ so that $\|x - y\| < \varepsilon/2$ and set $M = \|y\|_{L_\infty(\mu)}$. Define $x_M \in S$ by

$$x_M(t) = \begin{cases} x(t), & \text{if } |x(t)| \le M, \\ M \ \operatorname{sign} x(t), & \text{if } |x(t)| > M. \end{cases}$$

Observe that $|x - x_M| \le |x - y|$ a.e. Since $\lim_{\mu(A)\to 0} \|\mathbf{1}_A\| = 0$, there exists $\delta > 0$ so that for every $A \in \Omega$ with $\mu(A) < \delta$, $\|M\mathbf{1}_A\| < \varepsilon/2$. Thus for every $A \in \Omega$ with $\mu(A) < \delta$ we obtain

$$\|x\mathbf{1}_A\| \le \|x_M\mathbf{1}_A\| + \|(x - x_M)\mathbf{1}_A\| \le \|M\mathbf{1}_A\| + \|x - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

so the norm on E is absolutely continuous. □

Corollary 2.11. *Let E be a Köthe F-space on* $[0, 1]$ *with an absolutely continuous norm. Then E is separable. Moreover, the linear span D of simple functions supported on dyadic intervals is dense in E.*

Proof. Given any set $A \in \Sigma$, let (B_n) be a sequence of dyadic intervals so that $\lim_{n\to\infty} \mu(A \triangle B_n) = 0$. By the absolute continuity of the norm, $\|\mathbf{1}_A - \mathbf{1}_{B_n}\| = \|\mathbf{1}_{A\triangle B_n}\| \to 0$ as $n \to \infty$. Thus, D is dense in the set of all simple functions which, in turn, is dense in E by Proposition 2.10. □

We remark that the assumption of absolute continuity of the norm on the unit in the second part of Proposition 2.10 is essential, since $L_\infty(\mu)$ is an example of a Köthe F-space on (Ω, Σ, μ) in which the set of all simple functions is dense, however the norm is not absolutely continuous.

Theorem 2.12. *Let E be a Köthe F-space with an absolutely continuous norm on the unit on a finite atomless measure space (Ω, Σ, μ) and X an F-space. If* $\operatorname{dens} E(A) > \operatorname{dens} X$ *for every $A \in \Sigma^+$ then every operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Using Maharam's Theorem 1.14, we decompose $\Omega = \bigsqcup_{\alpha \in \mathcal{M}} \Omega_\alpha$ so that for every $\alpha \in \mathcal{M}$, the measure spaces $(\Omega_\alpha, \Sigma(\Omega_\alpha), \mu|_{\Sigma(\Omega_\alpha)})$ and $\varepsilon_\alpha \cdot (D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$ are isomorphic for some $\varepsilon_\alpha > 0$ such that $\sum_{\alpha \in \mathcal{M}} \varepsilon_\alpha = \mu(\Omega)$, where $\mathcal{M}$ is the Maharam set of the measure space (Ω, Σ, μ).

Fix any $A \in \Sigma^+$ and let $A_\alpha = A \cap \Omega_\alpha$ for each $\alpha \in \mathcal{M}$. Then fix any $\alpha \in \mathcal{M}$ with $\mu(A_\alpha) > 0$. By Maharam's theorem, the measure space $(A, \Sigma_{\omega_\alpha}(A), \mu_{\omega_\alpha}|_{\Sigma_{\omega_\alpha}(A)})$ is isomorphic to $(D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$. Hence, $E(A)$ contains a "Rademacher system" of cardinality $\aleph_\alpha$, that is, an ω_α-sequence of probabilistic independent mean zero signs $(r_\beta)_{\beta<\omega_\alpha}$. Since $\operatorname{dens} X < \aleph_\alpha$, there are indices $\beta \neq \gamma < \omega_\alpha$ such that $\|Th_\alpha\| < \mu(\Omega)^{-1}\varepsilon_\alpha \varepsilon$ where $h_\alpha = (r_\beta - r_\gamma)/2$. Observe that h_α is a sign on a subset $B_\alpha \subset A_\alpha$ of measure $\mu(B_\alpha) = \mu(A_\alpha)/2$. For convenience, we set $h_\alpha = 0$ and $B_\alpha = \emptyset$ if $\mu(A_\alpha) = 0$. By the absolute continuity of the norm on the unit, the series $h = \sum_{\alpha \in \mathcal{M}} h_\alpha$ converges in E, and h is a sign on the set $B = \bigsqcup_{\alpha \in \mathcal{M}} B_\alpha$ of measure

$$\mu(B) = \sum_{\alpha \in \mathcal{M}} \mu(B_\alpha) = \sum_{\alpha \in \mathcal{M}} \frac{\mu(A_\alpha)}{2} = \frac{\mu(A)}{2} .$$

Moreover, by the construction,

$$\|Th\| \le \sum_{\alpha \in \mathcal{M}} \|Th_\alpha\| < \mu(\Omega)^{-1}\varepsilon \sum_{\alpha \in \mathcal{M}} \varepsilon_\alpha = \varepsilon .$$

By Proposition 1.9, T is narrow. □

A similar argument allows to estimate the density of a Köthe F-space with an absolutely continuous norm on the unit.

Proposition 2.13. *Let E be a Köthe F-space on D^{ω_α} with an absolutely continuous norm on the unit. Then* $\operatorname{dens} E(A) \ge \aleph_\alpha$ *for each $A \in \Sigma^+_{\omega_\alpha}$. If, moreover, the norm of X is absolutely continuous then* $\operatorname{dens} E(A) = \aleph_\alpha$.

Proof. Suppose $A \in \Sigma^+_{\omega_\alpha}$. First we will show that $\operatorname{dens} E(A) \ge \aleph_\alpha$. Assume, on the contrary, that $\operatorname{dens} E(A) < \aleph_\alpha$. Observe that the identity operator on $E(A)$ is not narrow. Then using Proposition 1.9, we choose $\varepsilon > 0$ and $A_1 \in \Sigma_{\omega_\alpha}(A)^+$ so that for every $B \in \Sigma_{\omega_\alpha}(A_1)$, if $\mu_{\omega_\alpha}(B) \ge \mu_{\omega_\alpha}(A_1)/2$ then $\|x\| \ge \varepsilon$ for every sign x on B.

By Maharam's theorem, the measure space $(A_1, \Sigma_{\omega_\alpha}(A_1), \mu_{\omega_\alpha}|_{\Sigma_{\omega_\alpha}(A_1)})$ is isomorphic to $(D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$. Hence, $E(A_1)$ contains a "Rademacher system" of cardinality $\aleph_\alpha$, that is, an ω_α-sequence of probabilistic independent mean zero signs $(r_\beta)_{\beta<\omega_\alpha}$ on A_1. Since $\operatorname{dens} E(A_1) \le \operatorname{dens} E(A) < \aleph_\alpha$, there are indices $\beta_1 \neq \beta_2 < \omega_\alpha$ such that $\|x\| < \varepsilon$, where $x = (r_{\beta_1} - r_{\beta_2})/2$. Observe that x is a mean zero sign on some set $B \subseteq A_1$ with $\mu_{\omega_\alpha}(B) = \mu_{\omega_\alpha}(A_1)/2$, a contradiction.

Let the norm of E be absolutely continuous. We prove the converse inequality. By Proposition 2.10, the set $S(A)$ of all simple functions on A is dense in $E(A)$. Using the absolute continuity of the norm, one can show that the set $\widetilde{Z}(A)$ of all simple functions of the form $\sum_{k=1}^m q_k \mathbf{1}_{A\cap A_m}$, where $m \in \mathbb{N}$ and q_k are scalars from some countable dense subset Q of $\mathbb{K}$, and A_k are cylindric sets from Σ_{ω_α}, is dense in $S(A)$, and thus, in $E(A)$ (see Proposition 2.10).

It remains to observe that the set R of all cylindric sets from Σ_{ω_α} has cardinality $\aleph_\alpha$. Thus, $\operatorname{dens} E(A) \le |\widetilde{Z}(A)| \le \aleph_\alpha \cdot |Q| \cdot |R| = \aleph_\alpha$. □

Using the Maharam theorem, Theorem 2.12 can be reformulated as follows.

Corollary 2.14. *Let E be a Köthe F-space with an absolutely continuous norm on the unit on a finite atomless measure space (Ω, Σ, μ) with the Maharam set $\mathcal{M}$ and $\alpha = \min \mathcal{M}$. Let X be an F-space with* $\operatorname{dens} X < \aleph_\alpha$. *Then every operator $T \in \mathcal{L}(E, X)$ is narrow.*

Under a stronger assumption on X, we obtain a similar statement for strictly rich subspaces.

Theorem 2.15. *Suppose that E is a Köthe F-space on a finite atomless measure space (Ω, Σ, μ) for which there exists a reflexive Köthe–Banach space E_1 on (Ω, Σ, μ) with continuous inclusion embedding $E_1 \subseteq E$. Let $\mathcal{M}$ be the Maharam set of (Ω, Σ, μ) and $\alpha = \min \mathcal{M}$. Then the following assertions hold.*

(a) *Let X be a subspace E with H-codim $X < \aleph_\alpha^{\aleph_0}$. Then X is strictly rich.*

(b) *Let X be an F-space with H-dim $X < \aleph_\alpha^{\aleph_0}$. Then every operator $T \in \mathcal{L}(E, X)$ is strictly narrow.*

For the proof, we need the following lemma.

Lemma 2.16. *Let X be an infinite dimensional F-space. Then H-dim $X = |X| =$* $(\operatorname{dens} X)^{\aleph_0}$.

Proof of Lemma 2.16. For infinite dimensional F-spaces of continuum cardinality $\mathfrak{c}$ the equality H-dim $X = \mathfrak{c}$ was proved by Drewnowski in [34]. Since for infinite cardinal numbers $\kappa^2 = \kappa$, we get that if H-dim $X > \mathfrak{c}$ then

$$|X| = \sum_{n=1}^{\infty} \mathfrak{c} \cdot (\text{H-dim}\, X)^n = \sum_{n=1}^{\infty} \mathfrak{c} \cdot \text{H-dim}\, X = \aleph_0 \cdot \mathfrak{c} \cdot \text{H-dim}\, X = \text{H-dim}\, X\ .$$

Finally, the equality $|X| = (\operatorname{dens} X)^{\aleph_0}$ was proved by Kurochkin in [73]. □

Proof of Theorem 2.15. Observe first that (a) and (b) easily imply each other. So, for the real case we prove (a), and for the complex scalar case we prove (b).

The real case. Let X be a subspace satisfying the assumption of (a). We show that X satisfies condition (i) of Theorem 2.8. Let $A \in \Sigma^+$. Then by Lemma 2.16

$$\text{H-dim}\, L_\infty(A) = \big(\text{dens}\, L_\infty(A)\big)^{\aleph_0} \geq \big(\text{dens}\, E(A)\big)^{\aleph_0} \geq \aleph_\alpha^{\aleph_0} > \text{H-dim}\, X \, .$$

Thus, by Lemma 2.9, $X \cap L_\infty(A) \neq \{0\}$, and by Theorem 2.8, X is strictly rich.

The complex case. Let X be an F-space satisfying the assumption of (b), and let $T \in \mathcal{L}(E, X)$. Denote by $X_{\mathbb{R}}$ the F-space X considered as an F-space over the reals. We show that

$$\text{H-dim}\, X_{\mathbb{R}} < \aleph_\alpha^{\aleph_0} \, . \tag{2.2}$$

If X is finite dimensional then so is $X_{\mathbb{R}}$, and therefore, (2.2) holds. Let X be infinite dimensional, and let $(e_j)_{j \in J}$ be a Hamel basis of X. Then the linear span of the system $(e_j)_{j\in J} \cup (i \cdot e_j)_{j \in J}$ in $X_{\mathbb{R}}$ is $X_{\mathbb{R}}$, and hence, $\text{H-dim}\, X_{\mathbb{R}} \leq |J|^2 = |J| = \text{H-dim}\, X < \aleph_\alpha^{\aleph_0}$. Consider $\widetilde{E} = \{x \in E : \text{Im}\, x = 0\}$ and $\widetilde{E}_1 = \{x \in E_1 : \text{Im}\, x = 0\}$ as real spaces. Obviously, $\widetilde{E}$ is a real Köthe F-space and $\widetilde{E}_1$ is a real Köthe–Banach space on (Ω, Σ, μ). We show that $\widetilde{E}_1$ is reflexive using reflexivity of E_1 and the following well-known criterion: a Banach space Z is reflexive if and only if its unit ball B_Z is weakly compact. We know that B_{E_1} is compact with respect to the weak topology generated by functionals $f \in E_1^*$. Define a map $F : B_{E_1} \to B_{\widetilde{E}_1}$ by $F(x) = \text{Re}\, x$, and note that this map is onto. Since a continuous image of a compact set is compact, it is enough to show that F is $\sigma(E_1^*, \widetilde{E}_1^*)$-continuous. Let (x_α) be a net in B_{E_1} weakly converging to x. Our goal is to show that $(\text{Re}\, x_\alpha)$ converges to $\text{Re}\, x$ in the weak topology of $\widetilde{E}_1$. Indeed, given any $g \in \widetilde{E}_1^*$, we define $f \in E_1^*$ by $f(y) = g(\text{Re}\, y) + i g(\text{Im}\, y)$. Then

$$|g(\text{Re}\, x_\alpha) - g(\text{Re}\, x_\alpha)| = |g(\text{Re}(x_\alpha - x))| \leq |f(x_\alpha - x)| \to 0 \, .$$

Thus, we have proved that F is $\sigma(E_1^*, \widetilde{E}_1^*)$-continuous, and hence $\widetilde{E}_1$ is reflexive.

Obviously, $\widetilde{E}_1 \subseteq \widetilde{E}$ is a continuous inclusion. Hence, by (b) for the real case, the restriction $T|_{\widetilde{E}} : \widetilde{E} \to X_{\mathbb{R}}$ is a strictly narrow operator. Thus T is strictly narrow. □

Theorem 2.15 gives very restrictive conditions on the dimension of the range of an operator to be strictly narrow. However, under weaker assumptions an operator is narrow (see Theorem 2.12 and Corollary 2.14). Is it strictly narrow?

Open problem 2.17. Let E be a Köthe F-space with an absolutely continuous norm on (Ω, Σ, μ) and X be an F-space. Suppose that $\text{dens}\, E(A) > \text{dens}\, X$ for every $A \in \Sigma^+$. Does it follow that every operator $T \in \mathcal{L}(E, X)$ is strictly narrow?

After Lomonosov's elegant result that every compact operator $T \in \mathcal{L}(\ell_2)$ has a nontrivial invariant subspace, every class of "small" operators is checked whether it has the same property. We do not know of any results in this direction.

Open problem 2.18. Let E be a Köthe–Banach space on (Ω, Σ, μ). Does every narrow operator $T \in \mathcal{L}(E)$ have a nontrivial invariant subspace?

2.4 Narrow operators are compact on a suitable subspace

The following proposition shows that for every narrow operator there exists a subspace of the domain space so that the restriction of the operator to this subspace is compact.

Proposition 2.19. *Let E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ) with an absolutely continuous norm, and let X be a Banach space and $T \in \mathcal{L}(E, X)$ a narrow operator. Then for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists an atomless sub-σ-algebra Σ_1 of $\Sigma(A)$ such that the restriction $T_1 = T|_{E^0(\Sigma_1)}$ of T to the subspace $E^0(\Sigma_1) = \{x \in E(\Sigma_1) : \int_A x \, d\mu = 0\}$ is compact and $\|T_1\| \le \varepsilon$.*

For the proof we need a known fact asserting that an operator is compact and has "small" norm if it is sufficiently "small" at a Markushevich basis. A *Markushevich basis* (or just an *M-basis*) of a Banach space X is a *biorthogonal* system $(x_i, x_i^*)_{i \in I}$ of pairs $x_i \in X$, $x_i^* \in X^*$ with some index set I (i.e. $x_j^*(x_i) = \delta_{i,j}$) such that

(a) $(x_i)_{i \in I}$ is *complete* (or, in another terminology, *fundamental*), that is, $[x_i] = X$;

(b) *minimal*, i.e. $x_j \notin [x_i]_{i \in I \setminus \{j\}}$ for each $j \in I$;

(c) $(x_i^*)_{i \in I}$ is *total* in X^*, i.e. the w^*-closure of the linear span of (x_i^*) equals X^*.

Let X, Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is called *nuclear* if there exist sequences (x_n^*) in X^* and (y_n) in Y such that $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ and $Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for each $x \in X$. Every nuclear operator is compact [29, p. 170].

Lemma 2.20. *Let $(x_n, x_n^*)_{n=1}^{\infty}$ be an M-basis of a Banach space X, Y a Banach space, $T \in \mathcal{L}(X, Y)$, and $\varepsilon > 0$. If $\|Tx_n\| < \varepsilon_n$ for each $n \in \mathbb{N}$ where $\varepsilon_n = 2^{-n} \|x_n^*\|^{-1} \varepsilon$ then T is compact and $\|T\| \le \varepsilon$.*

The following elegant proof was communicated to us by Plichko.

Proof of Lemma 2.20. Fix any $n \in \mathbb{N}$. Observe that if $x = \sum_{k=1}^{n} a_k x_k$ and $\|x\| = 1$ then $\|x_k^*\| \ge |x_k^*(x)| = |a_k|$ for each $k = 1, \dots, n$. Thus

$$\|Tx\| \le \sum_{k=1}^{n} |a_k| \varepsilon_k \le \sum_{k=1}^{n} \|x_k^*\| \frac{\varepsilon}{2^k \|x_k^*\|} < \varepsilon ,$$

so $\|T\| \le \varepsilon$. And, since $\sum_{n=1}^{\infty} \|x_n^*\| \|Tx_n\| \le \varepsilon < \infty$, T is a kernel operator, and hence, is compact. □

Proof of Proposition 2.19. Fix any $\varepsilon > 0$ and $A \in \Sigma^+$. Using the definition of a narrow operator, we construct a normalized Haar-type system, beginning with the second term, $(g_n)_{n=2}^\infty$ with $\operatorname{supp} g_2 = A$ such that $\|Tg_n\| \le 2^{-n}(\mu(\operatorname{supp} g_n))^{-1}\|g_n\|\varepsilon$ for $n = 2, 3, \dots$ Let Σ_1 be the sub-σ-algebra of $\Sigma(A)$ generated by $(\operatorname{supp} g_n)_{n=2}^\infty$. By Proposition 2.10, the set of all simple functions is dense in E, and by absolute continuity of the norm, we obtain that the system $(g_n)_{n=1}^\infty$ is complete in $E(\Sigma_1)$ and hence, $[g_n]_{n=2}^\infty = E^0(\Sigma_1)$. Observe that the system $g_n^* = (\mu(\operatorname{supp} g_n))^{-1} g_n$, $n = 2, 3, \dots$ is biorthogonal to $(g_n)_{n=2}^\infty$ in $E^0(\Sigma_1)$. By Lemma 2.20, $T_1 = T|_{E^0(\Sigma_1)}$ is compact. □

In the case of a separable r.i. space, even more is true.

Theorem 2.21 ([110]). *Let E be an r.i. Banach space on $[0, 1]$ with an absolutely continuous norm, X a Banach space and $T \in \mathcal{L}(E, X)$ a narrow operator. Then for each $\varepsilon > 0$ there exists a subspace E_0 of E isometrically isomorphic to E such that the restriction $T|_{E_0}$ of T to E_0 is a compact operator with $\|T|_{E_0}\| \le \varepsilon$.*

Moreover, for each $\varepsilon > 0$ and each sequence of positive numbers $(\varepsilon_n)_{n=1}^\infty$ there exists a normalized Haar-type system $(g_n)_{n=1}^\infty$ in E such that $\|Tg_n\| < \varepsilon_n$ for $n = 1, 2, \dots$, and for the subspace $E_0 = [g_n]_{n=1}^\infty$ we have that $T|_{E_0}$ is a compact operator with $\|T|_{E_0}\| \le \varepsilon$.

As in the proof of Proposition 2.19, one can construct a sequence $(g_n)_{n=2}^\infty$ isometrically equivalent to the L_∞-normalized Haar system in E, beginning with the second term such that $\|Tg_n\|$ are small enough. However, this is not enough because of the absence of the first term. Therefore, our goal is to construct such a sequence, beginning with $n = 1$.

We need the following statement which follows easily from the definition of a narrow operator.

Lemma 2.22. *Let y be a mean zero sign on $A \in \Sigma^+$. Then for each $\varepsilon_1 > 0$ there exists a mean zero sign x on A such that $\|Tx\| < \varepsilon_1$ and which is independent of y, that is, for all sign numbers θ_1 and θ_2 one has $\mu\{t \in A : x(t) = \theta_1 \ \&\ y(t) = \theta_2\} = \mu(A)/4$.*

Proof of Proposition 2.21. Fix any $\varepsilon, \varepsilon_n > 0$ for $n = 1, 2, \dots$. Choose a mean zero sign $\overline{g}_1$ on $[0, 1]$ so that $\|T\overline{g}_1\| \le 2^{-1}\|\mathbf{1}_{[0,1]}\|\varepsilon$ and $\|T\overline{g}_1\| < \varepsilon_1$. Now we are going to construct a sequence $(\overline{g}_n)_{n=1}^\infty$ of signs such that for all $n = 1, 2, \dots$,

$$\|T\overline{g}_n\| \le 2^{-n}\big(\mu(\operatorname{supp}\overline{g}_n)\big)^{-1}\|\overline{g}_n\|\varepsilon \ \text{ and } \ \|T\overline{g}_n\| < \varepsilon_n\,, \tag{2.3}$$

and such that the sequence $(J\overline{g}_n)_{n=1}^\infty$ is isometrically equivalent to the L_∞-normalized Haar system in E, where $J \in \mathcal{L}(E)$ is the onto isometry defined by $Jx = g_1 \cdot x$, $x \in E$.

By Lemma 2.22, we choose a mean zero sign $\overline{g}_2$ on $[0, 1]$ independent of $\overline{g}_1$ so that $\|T\overline{g}_2\| \leq 2^{-2}\|\mathbf{1}_{[0,1]}\|\varepsilon$. To construct $\overline{g}_3$ and $\overline{g}_4$, let $A_{1,1} = \{t : \overline{g}_1(t) = 1\}$ and $A_{1,2} = \{t : \overline{g}_1(t) = -1\}$. Since $\overline{g}_2$ is independent of $\overline{g}_1$, we have that $f_{1,i} = \overline{g}_2 \cdot \mathbf{1}_{A_{1,i}}$ is a mean zero sign on $A_{1,i}$ for $i = 1, 2$. Again by Lemma 2.22, we choose a mean zero sign $\overline{g}_3$ on $A_{1,1}$ independent of $f_{1,1}$ so that $\|T\overline{g}_3\| < 2^{-3} \cdot 2\|\overline{g}_3\|\varepsilon$, and a mean zero sign $\overline{g}_4$ on $A_{1,2}$ independent of $f_{1,2}$ so that $\|T\overline{g}_4\| < 2^{-4} \cdot 2\|\overline{g}_4\|\varepsilon$. Likewise, to construct $\overline{g}_5, \overline{g}_6, \overline{g}_7$ and $\overline{g}_8$, we set $A_{2,1} = \{t \in A_{1,1} : \overline{g}_2(t) = 1\}$, $A_{2,2} = \{t \in A_{1,1} : \overline{g}_2(t) = -1\}$, $A_{2,3} = \{t \in A_{1,2} : \overline{g}_2(t) = -1\}$ and $A_{2,1} = \{t \in A_{1,2} : \overline{g}_2(t) = 1\}$. We remark that it is essential to define $A_{2,3}$ in such a way that $\overline{g}_2(t) = -1$ on this set, and $\overline{g}_2(t) = 1$ on $A_{2,4}$, because when operator J acts on the functions $\overline{g}_7$ and $\overline{g}_8$ which are being constructed, it reverses their signs. The independence of $\overline{g}_3$ of $f_{1,1}$ and $\overline{g}_4$ of $f_{1,2}$ yields that the functions

$$f_{2,1} = \overline{g}_3 \cdot \mathbf{1}_{A_{2,1}}, \quad f_{2,2} = \overline{g}_3 \cdot \mathbf{1}_{A_{2,2}}, \quad f_{2,1} = \overline{g}_4 \cdot \mathbf{1}_{A_{2,3}}, \quad f_{2,1} = \overline{g}_4 \cdot \mathbf{1}_{A_{2,4}}$$

are mean zero signs on the sets $A_{2,1}$, $A_{2,2}$, $A_{2,3}$ and $A_{2,4}$, respectively. By Lemma 2.22, we choose a mean zero sign $\overline{g}_5$ on $A_{2,1}$ independent of $f_{2,1}$ with $\|T\overline{g}_5\| < 2^{-5}2^2\|\overline{g}_5\|\varepsilon$, a mean zero sign $\overline{g}_6$ on $A_{2,2}$ independent of $f_{2,2}$ with $\|T\overline{g}_6\| < 2^{-6}2^2\|\overline{g}_6\|\varepsilon$, a mean zero sign $\overline{g}_7$ on $A_{2,3}$ independent of $f_{2,3}$ with $\|T\overline{g}_7\| < 2^{-7}2^2\|\overline{g}_7\|\varepsilon$ and a mean zero sign $\overline{g}_8$ on $A_{2,4}$ independent of $f_{2,4}$ with $\|T\overline{g}_8\| < 2^{-8}2^2\|\overline{g}_7\|\varepsilon$. Then we continue the construction in the same manner.

The sequence $(J\overline{g}_i)_{i=1}^{\infty}$ is isometrically equivalent to the L_∞-normalized Haar system $(\overline{h}_i)_{i=1}^{\infty}$ in E since, for each n, the sequences $(J\overline{g}_k)_{k=1}^{n}$ and $(\overline{h}_k)_{k=1}^{n}$ are equimeasurable, that is, for every choice of $\theta_i \in \{-1, 0, 1\}$,

$$\mu\{t : J\overline{g}_1(t) = \theta_1, \ldots, J\overline{g}_n(t) = \theta_n\} = \mu\{t : \overline{h}_1(t) = \theta_1, \ldots, \overline{h}_n(t) = \theta_n\}.$$

Thus $E_0 = [g_n]_{n=1}^{\infty}$ is isometrically isomorphic to E.

Normalizing $g_n = \frac{\overline{g}_n}{\|\overline{g}_n\|}$, by (2.3), we get that $\|Tg_n\| \leq 2^{-n}(\mu(\operatorname{supp} g_n))^{-1}\|g_n\|\varepsilon$ for all $n \in \mathbb{N}$. Since the system $g_n^* = (\mu(\operatorname{supp} g_n))^{-1}g_n$, $n = 1, 2, \ldots$ is biorthogonal to $(g_n)_{n=1}^{\infty}$, by Lemma 2.20, $T|_{E_0}$ is compact with $\|T|_{E_0}\| \leq \varepsilon$. □

Corollary 2.23. *Let E be an r.i. Banach space on* $[0, 1]$ *with an absolutely continuous norm, and let X be a rich subspace of E. Then for each $\varepsilon > 0$ there exists a subspace X_0 of X which is* $(1 + \varepsilon)$*-isomorphic to E and* $(1 + \varepsilon)$*-complemented in E.*

Proof. Let $T : E \to E/X$ be the quotient map. Since X is rich, T is narrow by definition. Fix any $\varepsilon > 0$. Using [79, Proposition 1.a.9] (more precisely, the well-known technique from there), we choose a sequence of positive numbers $(\varepsilon_n)_{n=1}^{\infty}$ such that if (x_n) is a normalized monotone basic sequence in a Banach space Z so that $[x_n]$ is 1-complemented in Z, and (y_n) is a sequence in Z with $\|x_n - y_n\| < \varepsilon_n$ for all $n = 1, 2, \ldots$, then the subspace $Y = [y_n]$ is $(1 + \varepsilon)$-isomorphic to $[x_n]$ and $(1 + \varepsilon)$-complemented in Z. By Theorem 2.21, there exists a sequence (g_n) in E

isometrically equivalent to the normalized Haar system in E such that $\|Tg_n\| < \varepsilon_n$ for each $n = 1, 2, \ldots$. Hence there is a sequence (f_n) in X such that $\|f_n - g_n\| < \varepsilon_n$ for each $n = 1, 2, \ldots$. By the above, $X_0 = [f_n]$ satisfies the desired properties. □

Using Corollary 2.23, we obtain that, under a minor assumption, a complemented rich subspace of E is isomorphic to E. Recall that an r.i. Banach space E is called s-concave with $1 \le s < \infty$ if there is a constant $M > 0$ such that for each $n \in \mathbb{N}$ and each $x_1, \ldots, x_n \in E$ we have

$$\Big(\sum_{k=1}^{n} \|x_k\|^s\Big)^{1/s} \le M \Big\| \Big(\sum_{k=1}^{n} |x_k|^s\Big)^{1/s} \Big\| .$$

The following statement is a consequence of Corollary 2.23 and [49, p. 240].

Proposition 2.24 ([110]). *Let E be an r.i. Banach space on $[0,1]$ that is s-concave for some $1 \le s < \infty$. Suppose that the Boyd index $p_E > 1$ and the Haar system is not equivalent to a disjoint sequence in E. Then every complemented rich subspace of E is isomorphic to E.*

Chapter 3

Some properties of narrow operators with applications to nonlocally convex spaces

In this chapter we apply narrow operator methods to study how small the space of operators is on nonlocally convex spaces.

The classical theorem of Day [28, p. 3] asserts that the space $L_p(\mu)$ for $0 < p < 1$ and an atomless measure μ has trivial dual space $L_p^*(\mu) = \{0\}$. Zabreiko [142] (1964) posed a question whether there exist nonzero compact operators on $L_p(\mu)$, $0 < p < 1$. This question was answered negatively in 1973, independently by Pallaschke [104] and Turpin [139]. Later Kalton [64] (1976) proved that in fact there do not exist nonzero compact operators from $L_p(\mu)$, $0 < p < 1$, to any topological vector space. Further, Kalton showed [67], see also [68], that there are no nonzero compact operators from E to any topological vector space X, for a fairly large class of r.i. F-spaces E which contains, in particular, all spaces having the following property (q)

$$\lim_{\mu(A)\to 0} \left\| \frac{\mathbf{1}_A}{\mu(A)} \right\| = 0.$$

Plichko and Popov [110] and Popov [115] gave a very short proof that there are no nonzero narrow operators from absolutely continuous Köthe F-spaces with property (q) to any F-space (see Theorem 3.5).

Definition 3.1. A Köthe F-space will be called *strongly nonconvex* if it satisfies property (q).

Every strongly nonconvex Köthe F-space E has trivial dual $E^* = \{0\}$ [122, p. 194]. Notice that $L_p(\mu)$-spaces with $0 < p < 1$ and an atomless measure μ are strongly nonconvex. The following statement (the proof of which is straightforward) gives more examples of strongly nonconvex spaces.

Proposition 3.2. *Let (Ω, Σ, μ) be a finite atomless measure space, $\Omega = \Omega_1 \sqcup \Omega_2$ and E_i be strongly nonconvex Köthe F-spaces on $(\Omega_i, \Sigma(\Omega_i), \mu|_{\Omega_i})$, $i = 1, 2$. Then for each $p \in (0, \infty]$ the space $E = E_1 \oplus_p E_2$ is a strongly nonconvex Köthe F-space on (Ω, Σ, μ). If, moreover, E_1 and E_2 have absolutely continuous norms then so does E.*

Clearly every strongly nonconvex Köthe F-space has absolutely continuous norm on the unit. Thus, by Proposition 2.10 we obtain the following statement.

Corollary 3.3. *A strongly nonconvex Köthe F-space E has absolutely continuous norm if and only if the set of all simple functions is dense in E.*

However there do exist strongly nonconvex Köthe F-spaces whose norm is not absolutely continuous as demonstrated by the following example.

Example 3.4 ([69]). Let $0 < p < 1$. Then the Köthe F-space $E = (\sum_{n=1}^{\infty} L_p)_\infty$ on $[0, 1]$ is strongly nonconvex, however, the norm on E is not absolutely continuous.

Proof. To consider a concrete representation of E, we decompose $[0,1] = \bigsqcup_{n=1}^{\infty} A_n$ with $A_n \in \Sigma^+$. Let $E = \{x \in L_0 : \|x\| \overset{\text{def}}{=} \sup_n (\int_{A_n} |x|^p \, \mathrm{d}\mu)^{\frac{1}{p}} < \infty\}$. It is clear that E is a Köthe F-space on $[0, 1]$. Let $x \in E$ with $\int_{A_n} |x|^p \, \mathrm{d}\mu = 1$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|x \cdot \mathbf{1}_{A_n}\| = 1$. Since $\lim_{n\to\infty} \mu(A_n) = 0$, this implies that the F-norm of E is not absolutely continuous.

It remains to show that E is strongly nonconvex. Fix any $\varepsilon > 0$, and set $\delta = \varepsilon^{\frac{1}{1-p}}$. For any $A \in \Sigma$ with $\mu(A) < \delta$ we have for each $n \in \mathbb{N}$

$$\int_{A_n} \left| \frac{\mathbf{1}_A}{\mu(A)} \right|^p \mathrm{d}\mu = \frac{\mu(A \cap A_n)}{\mu(A)^p} \le \mu(A)^{1-p} < \delta^{1-p} = \varepsilon \,.$$

Thus, $\|\frac{\mathbf{1}_A}{\mu(A)}\| < \varepsilon$ whenever $\mu(A) < \delta$. □

3.1 Nonexistence of nonzero narrow operators

In this section we use a method of narrow operators to give a very short proof of the nonexistence of nonzero "small" operators on strongly nonconvex spaces.

Theorem 3.5 ([110, 115]). *Let E be a strongly nonconvex Köthe F-space with an absolutely continuous norm, and let X be any F-space. If $T \in \mathcal{L}(E, X)$ is a narrow operator then $T = 0$.*

Proof. By Corollary 3.3, since simple functions are dense in E, it is enough to prove that $T\mathbf{1}_A = 0$ for every $A \in \Sigma^+$. Given any $A \in \Sigma^+$, by Lemma 1.11 there exists a sequence (h_n) in E such that $\lim_n \|Th_n\| = 0$, $h_n = \mathbf{1}_{A'_n} - (2^n - 1)\mathbf{1}_{A''_n}$, $A'_n \sqcup A''_n = A$ and $\mu(A''_n) = 2^{-n}\mu(A_n)$. Then $\lim_{n\to\infty} \|\mathbf{1}_A - h_n\| = \lim_{n\to\infty} \|2^n \mathbf{1}_{A''_n}\| = \lim_{n\to\infty} 2^{n(p-1)} \mu(A) = 0$. Since $\mathbf{1}_A = \lim_{n\to\infty} h_n$ we obtain that $\|T\mathbf{1}_A\| = \lim_{n\to\infty} \|Th_n\| = 0$. □

The same proof gives for a more precise result.

Theorem 3.6 ([69]). *Let E be a strongly nonconvex Köthe F-space, and X be an F-space. If $T \in \mathcal{L}(E, X)$ is a narrow operator then $Tx = 0$ for every essentially bounded element $x \in E$.*

Corollary 3.7. *Let E be a strongly nonconvex Köthe F-space with an absolutely continuous norm, and let X be an F-space. If $T \in \mathcal{L}(E, X)$ is an AM-compact operator then $T = 0$.*

In particular, for $E = L_p(\mu)$ with $0 < p < 1$ we obtain the following statements.

Corollary 3.8. *Let (Ω, Σ, μ) be a finite atomless measure space, $0 < p < 1$ and X be an F-space.*

(1) *If $T \in \mathcal{L}(L_p(\mu), X)$ is a narrow operator then $T = 0$.*

(2) *If $T \in \mathcal{L}(L_p(\mu), X)$ is an AM-compact operator then $T = 0$.*

3.2 The separable quotient space problem

In the first edition of his book [122] (1972) Rolewicz asked whether there exists an infinite dimensional F-space with no separable infinite dimensional quotient space. This question was answered in [112] using techniques of narrow operators. Indeed, the following immediate consequence of Theorem 3.5 and Corollary 2.14 implies, in particular, that the space $L_p\{-1, 1\}^{\omega_\alpha}$ with $\alpha > 0$ has no separable quotient space.

Corollary 3.9 ([110, 112]). *Let E be a strongly nonconvex Köthe F-space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ) with the Maharam set $\mathcal{M}$ and $\alpha = \min \mathcal{M}$. Let X be an F-space with* $\operatorname{dens} X < \aleph_\alpha$. *Then $\mathcal{L}(E, X) = \{0\}$. Hence, the density of every nontrivial quotient space E/Y where Y is a subspace of E satisfies* $\operatorname{dens} E/Y \geq \aleph_\alpha$. *In particular, this holds for $L_p(\mu)$ if $0 < p < 1$.*

See also [136] and [137] for new F-spaces without separable infinite dimensional quotient spaces. We note, that it is still unknown whether there exists an infinite dimensional Banach space with no separable infinite dimensional quotient space (see, for example, [76, 137]).

3.3 Isomorphic classification of strongly nonconvex Köthe F-spaces

The aim of this section is to prove the following theorem.

Theorem 3.10. *Let E_i be strongly nonconvex Köthe F-spaces with absolutely continuous norms on finite atomless measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ with the Maharam sets $\mathcal{M}_i$, respectively. If E_1 and E_2 are isomorphic then $\mathcal{M}_1 = \mathcal{M}_2$.*

In particular, for $L_p(\mu)$-spaces with $0 < p < 1$ this fact implies more.

Theorem 3.11 ([113, 110]). *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ be finite atomless measure spaces with the Maharam sets $\mathcal{M}_i$ and $0 < p < 1$. Then the following assertions are equivalent:*

(i) $L_p(\mu_1)$ *and* $L_p(\mu_2)$ *are isomorphic.*

(ii) $L_p(\mu_1)$ *and* $L_p(\mu_2)$ *are isometrically isomorphic.*

(iii) $\mathcal{M}_1 = \mathcal{M}_2$.

Implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

For the proof of Theorem 3.10, we need some auxiliary statements.

Lemma 3.12. *Let X and Y be F-spaces, $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ and suppose that $\mathcal{L}(X_2, Y_1) = \{0\}$. Then $TX_2 \subseteq Y_2$ for every $T \in \mathcal{L}(X, Y)$.*

Proof. We set $\pi : Y \to Y$, $\pi(y_1 + y_2) = y_1$, where $y_1 \in Y_1$, $y_2 \in Y_2$. Then for every $T \in \mathcal{L}(X, Y)$ the operator $S : X_2 \to Y_1$ defined by $Sx = \pi(Tx)$, for each $x \in X_2$ is linear and continuous. Hence, by the assumption, $S = 0$, that is $Tx \in Y_2$ for each $x \in X_2$. □

Lemma 3.12 has the following two immediate consequences.

Corollary 3.13. *Let X and Y be F-spaces, $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, $\mathcal{L}(X_2, Y_1) = \{0\}$ and $\mathcal{L}(Y_2, X_1) = \{0\}$. Then $TX_2 = Y_2$ for every onto isomorphism $T \in \mathcal{L}(X, Y)$.*

Corollary 3.14. *Let X and Y be F-spaces such that $\mathcal{L}(X, Y) = \{0\}$, and let $Z = X \oplus Y$. Then $TX = X$ for every automorphism $T : Z \to Z$.*

The following statement easily follows from the definition of a narrow operator.

Lemma 3.15. *Let X be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ), $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$ be any partition into measurable subsets, and Y be an F-space. Then an operator $T \in \mathcal{L}(X, Y)$ is narrow if and only if for every $n \in \mathbb{N}$ the operator $T_n = T|_{X(\Omega_n)}$ is narrow.*

The main tool for the proof of Theorem 3.10 is the following statement.

Proposition 3.16. *Let E_i be Köthe F-spaces with absolutely continuous norms on finite atomless measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ with the Maharam sets $\mathcal{M}_i$, respectively, $i = 1, 2$. Assume, in addition, that E_1 has an absolutely continuous norm on the unit, and $\beta < \alpha$ for each $\alpha \in \mathcal{M}_1$ and $\beta \in \mathcal{M}_2$. Then $\mathcal{L}(E_1, E_2) = \{0\}$.*

Proof. Fix $T \in \mathcal{L}(E_1, E_2)$. For every $\beta \in \mathcal{M}_2$ let T_β be the continuous linear operator $T_\beta : E_1 \to E_2(D^{\omega_\beta})$ defined by $T_\beta x = Tx \cdot \mathbf{1}_{D^{\omega_\beta}}$. By Corollary 2.14, T_β is narrow. By Theorem 3.5, we obtain $T_\beta = 0$ for each $\beta \in \mathcal{M}_Y$. Thus $T = 0$. □

Now we are ready to prove the main result.

Proof of Theorem 3.10. Using Maharam's theorem, for $i = 1, 2$, we decompose $\Omega_i = \bigsqcup_{\beta \in \mathcal{M}_i} \Omega_{i,\beta}$ such that $(\Omega_{i,\beta}, \Sigma(\Omega_{i,\beta}), \mu_{\Sigma(\Omega_{i,\beta})})$ is isomorphic to $\varepsilon_{i,\beta} \cdot D^{\omega_\beta}$ for every $\beta \in \mathcal{M}_i$ and some $\varepsilon_{i,\beta} > 0$.

Let $T \in \mathcal{L}(E_1, E_2)$ be an isomorphism and let α be an ordinal. We set

$$A_i = \bigsqcup_{\mathcal{M}_i \ni \beta \leq \alpha} \Omega_{i,\beta}, \quad B_i = \bigsqcup_{\mathcal{M}_i \ni \beta > \alpha} \Omega_{i,\beta}, \quad C_i = \bigsqcup_{\mathcal{M}_i \ni \beta < \alpha} \Omega_{i,\beta}, \quad D_i = \bigsqcup_{\mathcal{M}_i \ni \beta \geq \alpha} \Omega_{i,\beta}$$

for $i = 1, 2$. By Proposition 3.16,

$$\begin{aligned} \mathcal{L}\big(E_1(B_1), E_2(A_2)\big) = \mathcal{L}\big(E_2(B_2), E_1(A_1)\big) = \mathcal{L}\big(E_1(D_1), E_2(C_2)\big) \\ = \mathcal{L}\big(E_2(D_2), E_1(C_1)\big) = \{0\} . \end{aligned}$$

By Corollary 3.13, $TE_1(B_1) = E_2(B_2)$ and $TE_1(D_1) = E_2(D_2)$. Thus, $B_1 \neq D_1$ if and only if $B_2 \neq D_2$. In other words, $\alpha \in \mathcal{M}_1$ if and only if $\alpha \in \mathcal{M}_2$. □

Chapter 4

Noncompact narrow operators

In this chapter we consider how "large" narrow operators can be. In previous chapters we saw that all compact operators are narrow, but here we consider the question whether the converse is true.

It turns out that on most Köthe–Banach spaces E on a finite atomless measure space (Ω, Σ, μ), there do exist noncompact narrow operators. In fact even conditional expectation operators may be narrow. Surprisingly, even operators which act as isomorphisms on certain subspaces may still be narrow. To make this precise we need the following definitions.

Definition 4.1. Let X, Y, Z be Banach spaces. We say that an operator $T \in \mathcal{L}(X, Y)$ *fixes a copy of* Z provided there exists a subspace X_0 of X isomorphic to Z such that the restriction $T|_{X_0}$ of T to X_0 is an isomorphic embedding. Otherwise we say that T is Z*-strictly singular*.

Definition 4.2. An operator $T \in \mathcal{L}(X)$ is called an *Enflo operator* if T fixes a copy of X.

The name "Enflo operator" is due to the following famous Enflo's theorem on primarity of L_p: if the space L_p, $1 \leq p < \infty$, is decomposed into a direct sum of closed subspaces $L_p = X \oplus Y$ then, at least, one of X, Y is isomorphic to L_p (see [37] for the case $p = 1$, [94] (1974) for other values of p and [80, p. 179] for a more general setting of r.i. spaces). Equivalently, if the identity of L_p is a sum of two projections $I = P + Q$ then, at least, one of P, Q is an Enflo operator. This equivalence can be obtained using Pełczyński's decomposition method [80, p. 54] and the fact that every subspace X of L_p that is isomorphic to L_p contains a further subspace $Y \subseteq X$ isomorphic to L_p and complemented in L_p [49, p. 239].

This chapter is devoted to the study of the following questions.

Problem 4.3. Does there exist a noncompact narrow operator $T \in \mathcal{L}(E)$?

Problem 4.4. Does there exist an Enflo narrow operator $T \in \mathcal{L}(E)$?

Problem 4.5. Suppose there exists a noncompact operator $T \in \mathcal{L}(E, X)$. Does there exist a noncompact narrow operator $T \in \mathcal{L}(E, X)$?

Problem 4.6. For what sub-σ-algebras $\mathcal{F}$ of an atomless σ-algebra Σ is the conditional expectation operator $M^{\mathcal{F}}$ on $L_1(\Omega, \Sigma, \mu)$ narrow or strictly narrow?

In Sections 4.1 and 4.2 we show that the answers to Problems 4.3–4.5 are affirmative, under mild restrictions on the space E. In Section 4.3 we answer Problem 4.6 and give full characterization of such sub-σ-algebras $\mathcal{F}$.

4.1 Conditional expectation operators with respect to purely atomic sub-σ-algebras

We start from a simple example of a noncompact strictly narrow operator.

Proposition 4.7. *Let E be a Köthe–Banach space on (Ω, Σ, μ) and let $\mathcal{F}$ be a purely atomic sub-σ-algebra of Σ with the atoms $(A_i)_{i\in I}$. If the conditional expectation operator*

$$M^{\mathcal{F}} x = \sum_{i\in I}\Big(\frac{1}{\mu(A_i)}\int_{A_i} x \,\mathrm{d}\mu\Big)\cdot \mathbf{1}_{A_i}$$

is well defined on E and bounded, then $M^{\mathcal{F}}$ is strictly narrow. Moreover, if I is infinite, then $M^{\mathcal{F}}$ is noncompact.

The proof is obvious.

Note that $M^{\mathcal{F}}$ is well defined and has norm one on any r.i. space E. However, this is not always the case for Köthe–Banach spaces, as the following example shows.

Example 4.8. There exists a Köthe–Banach space E on $[0,1]$, and a purely atomic sub-σ-algebra $\mathcal{F}$ of Σ such that the conditional expectation operator $M^{\mathcal{F}}$ is not well defined on E.

Proof. Consider the Köthe–Banach space on $[0,1]$ defined by $E = L_1[0,1/2] \oplus_\infty L_\infty[1/2,1]$. Let (a_n) and (b_n) be sequences of positive numbers such that

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2}, \quad \sum_{n=1}^{\infty} a_n b_n = 1 \quad \text{and} \quad \lim_{n\to\infty} a_n = \infty\,.$$

Decompose $[0,1/2] = \bigsqcup_{n=1}^{\infty} B_n$ and $[1/2,1] = \bigsqcup_{n=1}^{\infty} C_n$ with $B_n, C_n \in \Sigma$ such that $\mu(B_n) = \mu(C_n) = b_n$ for each n. Set $A_n = B_n \sqcup C_n$ for each $n = 1, 2, \dots$, and denote by $\mathcal{F}$ the σ-algebra generated by the atoms $(A_n)_{n=1}^{\infty}$. Let $x = \sum_{n=1}^{\infty} a_n \mathbf{1}_{B_n}$. Then $x \in E$, $\|x\| = 1$, and the function $M^{\mathcal{F}} x = \frac{1}{2}\sum_{n=1}^{\infty} a_n \mathbf{1}_{A_n}$ is unbounded on $[1/2,1]$ and hence, does not belong to E. □

Using the same idea and replacing ∞ with any number $p \in (1,+\infty)$ in Example 4.8, one can construct a Köthe–Banach space E on $[0,1]$ with an absolutely continuous norm and a conditional expectation operator with respect to a purely atomic sub-σ-algebra which is not well defined on E.

One can show, that if a Köthe–Banach space E has an absolutely continuous norm and the operator $M^{\mathcal{F}}$ is well defined in E then it is AM-compact. In the next section we will construct a strictly narrow operator which is not AM-compact.

4.2 A strictly narrow projection from E onto a subspace E_0 isometrically isomorphic to E

The main result of this section asserts that every r.i. Banach space E has a complemented subspace E_0 isometrically isomorphic to E such that there exist two projections from E onto E_0 one of which is strictly narrow and the other is not narrow. This answers Problem 4.4. Also, it follows that the property of an operator to be narrow is not a property of its image.

The proof is quite involved, and therefore we prove this theorem first for the case of spaces defined on the square $[0,1]^2$, and then for a general measure space.

This section is organized as follows. First we study the example of the operator of integration with respect to the second variable.

Example 4.9. Consider $L_1([0,1]^2, \sigma(\Sigma \times \Sigma), \lambda \times \lambda)$, where λ is the Lebesgue measure on the Lebesgue σ-algebra Σ on $[0,1]$. Let $\mathcal{G} = \Sigma \times \{[0,1]\}$ be the sub-σ-algebra of the Lebesgue σ-algebra on $[0,1]^2$. Note that the conditional expectation operator $M^{\mathcal{G}}$ is equal to

$$M^{\mathcal{G}} x(s,t) = \int_{[0,1]} x(s,t')\,\mathrm{d}t'$$

for each $x \in L_1[0,1]^2$.

We prove that $M^{\mathcal{G}}$ is a strictly narrow operator in $L_1[0,1]^2$ (Theorem 4.10). This implies that, if $M^{\mathcal{G}}$ is well defined and bounded on a Köthe–Banach space E on $[0,1]^2$ then it is strictly narrow. In particular, $M^{\mathcal{G}}$ is strictly narrow on any r.i. Banach space E on $[0,1]^2$.

Next we prove that for an r.i. Banach space E on $[0,1]^2$ the subspace E_0, which consists of all functions which do not depend of the second coordinate, is isometrically isomorphic to E (Proposition 4.12). Clearly E_0 is the range of $M^{\mathcal{G}}$.

In Theorem 4.13 we construct a projection from E onto E_0 which is not narrow. This ends the case of spaces defined on $[0,1]^2$.

The general case is proved in Theorem 4.17.

Results of this section were obtained in several papers. Theorem 4.10 was proved in [110] (1990), the example constructed in Theorem 4.13 comes from [118] (2002). The fact that $M^{\mathcal{G}}$ is strictly narrow on L_p, $1 \le p \le \infty$ was first proved in [71] (2009), and that it is strictly narrow in any Köthe space on $[0,1]$ on which it is bounded was proved in [31] (2008).

The case of spaces on [0, 1]

Our first goal is to prove the following result.

Theorem 4.10. *The conditional expectation operator $M^{\mathcal{G}}$ is a strictly narrow operator on $L_1[0,1]^2$.*

We remark that the proof that $M^{\mathcal{G}}$ is narrow is much easier.

For the proof of Theorem 4.10, we need the following lemma.

Lemma 4.11. *Let $A \subseteq [0,1]^2$ be any measurable subset and for any $s \in [0,1]$ let $A_s = \{t \in [0,1] : (s,t) \in A\}$. Then there exists a measurable function $\varphi_A : [0,1] \to [0,1]$ such that for almost all $s \in [0,1]$*

$$\lambda\Big(A_s \cap \big[0, \varphi_A(s)\big]\Big) = \frac{\lambda(A_s)}{2}\,. \tag{4.1}$$

Proof. Since $A \subseteq [0,1]^2$ is measurable, using Fubini's theorem and standard arguments, one can show that the set A_s is measurable for almost all $s \in [0,1]$ (in the sequel, for simplicity of the notation we consider these values of s only). We set

$$M_s = \Big\{t \in [0,1] : \lambda\Big(A_s \cap \big[0, t\big]\Big) = \frac{\lambda(A_s)}{2}\Big\}\,.$$

It is easy to show that for almost all values of s the set M_s is closed and nonempty. We define $\varphi_A(s) = \max M_s$. □

Proof of Theorem 4.10. Let $A \subseteq [0,1]^2$ be a measurable set. By Lemma 4.11, there exists a function $\varphi_A : [0,1] \to [0,1]$ such that (4.1) holds. Define

$$x(s,t) = \begin{cases} 1, & \text{if } (s,t) \in A \text{ and } t \le \varphi_A(s); \\ -1, & \text{if } (s,t) \in A \text{ and } t > \varphi_A(s); \\ 0, & \text{if } (s,t) \notin A. \end{cases}$$

Clearly, $x^2 = \mathbf{1}_A$. Moreover,

$$\int_{[0,1]^2} x(s,t)\,ds\,dt = \lambda\big\{(s,t) \in A : t \le \varphi_A(s)\big\} - \lambda\big\{(s,t) \in A : t > \varphi_A(s)\big\}$$

and

$$\begin{aligned} \lambda\big\{(s,t) \in A : t \le \varphi_A(s)\big\} &= \int_{[0,1]} \lambda\Big(A_s \cap \big[0, \varphi_A(s)\big]\Big) ds \\ &= \int_{[0,1]} \frac{\lambda(A_s)}{2} ds = \frac{1}{2} \int_{[0,1]} \lambda(A_s) ds = \frac{1}{2}\lambda(A). \end{aligned}$$

Hence $\lambda\{(s,t) \in A : t > \varphi_A(s)\} = \lambda(A)/2$, and thus

$$\int_{[0,1]^2} x(s,t)\,ds\,dt = 0\,.$$

Therefore, we obtain for almost every $(s,t) \in [0,1]^2$

$$
\begin{aligned}
M^{\mathcal{G}} x(s,t) = \int_0^1 x(s,t')\,\mathrm{d}t' &= \int_{[0,\varphi_A(s)]} x(s,t')\,\mathrm{d}t' + \int_{[\varphi_A(s),1]} x(s,t')dt' \\
&= \int_{[0,\varphi_A(s)]\cap A_s} dt' - \int_{[\varphi_A(s),1]\cap A_s} dt' \\
&= \lambda\Big([0,\varphi_A(s)] \cap A_s\Big) - \lambda\Big([\varphi_A(s),1] \cap A_s\Big) \\
&= \frac{1}{2}\lambda(A) - \frac{1}{2}\lambda(A) = 0\,.
\end{aligned}
$$

□

Proposition 4.12. *Let E be an r.i. Banach space on $[0,1]^2$. Then the subspace E_0 consisting of all functions which do not depend on the second coordinate is isometrically isomorphic to E.*

Proof. First, we define the r.i. space $E[0,1]$ on $[0,1]$ as follows. $E[0,1]$ consists of all $\widetilde{x} \in L_1[0,1]$ for which there exists $x \in E$ such that x and $\widetilde{x}$ are equimeasurable in modulus, that is,

$$\lambda\{t \in [0,1] : |\widetilde{x}(t)| < u\} = \mu\{(s,t) \in [0,1]^2 : |x(s,t)| < u\},$$

for every $u > 0$, endowed with the norm $\|\widetilde{x}\| = \|x\|$.

Let $x_0 \in E_0$. Then, by the definition of E_0, there exists a function $\widetilde{x} : [0,1] \to \mathbb{K}$ such that for almost all $s \in [0,1]$ and $t \in [0,1]$ $x(s,t) = \widetilde{x}(s)$. We claim that x and $\widetilde{x}$ are equimeasurable in modulus. Indeed, for all $u > 0$

$$\{(s,t) \in [0,1]^2 : |x(s,t)| < u\} = \{s \in [0,1] : |\widetilde{x}(s)| < u\} \times [0,1],$$

up to a set of measure zero. Thus, $\widetilde{x} \in E[0,1]$ and $\|\widetilde{x}\|_{E[0,1]} = \|x\|_E$. Hence E_0 and $E[0,1]$ are isometric. Since measure spaces $[0,1]$ and $[0,1]^2$ are isomorphic and are of the same measure, we conclude that $E[0,1]^2$ is isometric with $E[0,1]$ and thus also with E_0. □

We note that by Example 4.9, $E_0 = M^{\mathcal{G}}(E)$.

Theorem 4.13. *Let E be an r.i. Banach space on $[0,1]^2$. Then there exists a nonnarrow projection Q of E onto $E_0 = M^{\mathcal{G}}(E)$. Moreover, Q has the following property: there exists a decomposition $[0,1]^2 = C \sqcup D$ such that the restrictions $Q|_{E(C)}$ and $Q|_{E(D)}$ are isomorphic embeddings.*

For the proof we need the following easy description of all complements to a complemented subspace of a Banach space (actually, we will need only one implication of this description).

Proposition 4.14. *Suppose that a Banach space X is decomposed into a direct sum of subspaces $X = Y \oplus Z$ with the projection P of X onto Y parallel to Z (i.e. with* $\ker P = Z$*). Let $T \in \mathcal{L}(Z, Y)$. Then $X = Y \oplus Z_1$, where $Z_1 = \{z + Tz : z \in Z\}$, is another decomposition and the projection $Q = P - T(I - P)$ of X onto Y is parallel to Z_1. Moreover, $Z_1 \neq Z$ if $T \neq 0$.*

Conversely, if $X = Y \oplus Z_1$ for some subspace Z_1 of X then there exists a unique operator $T \in \mathcal{L}(Z, Y)$, such that $Z_1 = \{z + Tz : z \in Z\}$.

Below we provide a sketch of the proof.

Sketch of the proof of Proposition 4.14. One can easily verify that $Q^2 = Q$, $QY \subseteq Y$, $Qy = y$ for each $y \in Y$, and $\ker Q = Z_1$. Thus Q is a projection of X onto Y parallel to Z_1.

If $T \neq 0$ then $Q \neq P$ and hence, $Z_1 \neq Z$.

Let $X = Y \oplus Z_1$ be another decomposition. Then $T = -Q|_Z$, where Q is the projection of X onto Y parallel to Z_1, is the desired operator. The uniqueness of T follows from the first part of the proposition. □

Proof of Theorem 4.13. Let $E_1 = \ker M^{\mathcal{G}}$, $C = [0, 1/2) \times [0, 1]$, $D = [1/2, 1] \times [0, 1]$, $E_i^C = \{\mathbf{1}_C \cdot x : x \in E_i\}$ and $E_i^D = \{\mathbf{1}_D \cdot x : x \in E_i\}$ for $i = 0, 1$. Observe that $E_i = E_i^C \oplus E_i^D$ for $i = 0, 1$. Since $E(C) \simeq E_0^C \simeq E_0^D$ (here $\simeq$ means that the spaces are isometrically isomorphic) and E_1^C is a subspace of $E(C)$, there exists an isometric embedding $T^C : E_1^C \to E_0^D$. Analogously, there exists an isometric embedding $T^D : E_1^C \to E_0^C$. Thus the operator $T : E_1 \to E_0$ defined by $T = T^C \oplus T^D$ is an isometric embedding such that $TE_1^C \subseteq E_0^D$ and $TE_1^D \subseteq E_0^C$. Then, by Proposition 4.14, the operator $Q = M^{\mathcal{G}} - T(I - M^{\mathcal{G}})$ is a projection from E onto E_0. We claim that Q satisfies the desired properties.

Indeed, let $x \in E(C)$. Since $\mathbf{1}_C \cdot M^{\mathcal{G}} x = M^{\mathcal{G}} x$ and $TE_1^C \subseteq E_0^D$, we have that

$$\mathbf{1}_C \cdot Qx = \mathbf{1}_C \cdot M^{\mathcal{G}} x - \mathbf{1}_C \cdot T(I - M^{\mathcal{G}})x = M^{\mathcal{G}} x$$

and

$$\mathbf{1}_D \cdot Qx = \mathbf{1}_D \cdot M^{\mathcal{G}} x - \mathbf{1}_D \cdot T(I - M^{\mathcal{G}})x = T(I - M^{\mathcal{G}})\, x\, .$$

Hence,

$$\begin{aligned}\|Qx\| &\geq \max\Big\{\|\mathbf{1}_C \cdot Qx\|, \|\mathbf{1}_D \cdot Qx\|\Big\} = \max\Big\{\|M^{\mathcal{G}} x\|, \|T(I - M^{\mathcal{G}})x\|\Big\} \\ &= \max\Big\{\|M^{\mathcal{G}} x\|, \|(I - M^{\mathcal{G}})\, x\|\Big\} \geq \frac{\|M^{\mathcal{G}} x\| + \|(I - M^{\mathcal{G}})x\|}{2} \geq \frac{\|x\|}{2}.\end{aligned}$$

Thus $Q|_{E(C)}$ is an isomorphic embedding. Analogously, $Q|_{E(D)}$ is an isomorphic embedding. In particular, Q is not narrow. □

Theorems 4.10 and 4.13 imply the following result.

Corollary 4.15. *Let E be an r.i. Banach space on $[0,1]^2$. Then there exists a complemented subspace E_0 of E isometrically isomorphic to E, and two decompositions $E = E_0 \oplus E_1$ and $E = E_0 \oplus E_2$ into (closed) subspaces, such that E_1 is strictly rich and E_2 is not rich.*

The case of general measure spaces

By the Carathéodory theorem, analogs of Theorems 4.10, and 4.13 and Proposition 4.12 hold for any separable atomless probability space. The most natural way is to reformulate them for spaces on the unit interval.

Corollary 4.16. *There exists a sub-σ-algebra $\mathcal{F}$ of the Lebesgue σ-algebra Σ on $[0,1]$ such that the following conditions hold.*

(i) *The conditional expectation operator $M^{\mathcal{F}}$ is a strictly narrow projection on L_1. Consequently, $M^{\mathcal{F}}$ is a strictly narrow projection on any Köthe–Banach space E on $[0,1]$ whenever $M^{\mathcal{F}}$ is well defined and bounded on E.*

(ii) *For any r.i. Banach space E on $[0,1]$, $M^{\mathcal{F}}$ is a strictly narrow projection from E onto a subspace isometrically isomorphic to E.*

(iii) *For every r.i. Banach space E on $[0,1]$ there exists a nonnarrow projection Q from E onto $M^{\mathcal{F}}(E)$, and, in addition, Q has the following property: there exists a decomposition $[0,1] = \Omega_1 \oplus \Omega_2$ with $\Omega_i \in \Sigma^+$ such that the restrictions $Q|_{E(\Omega_i)}$ are isomorphic embeddings for $i = 1, 2$.*

The remainder of this section is devoted to proving the generalization of Theorems 4.10, and 4.13 and Proposition 4.12 to spaces of functions defined on arbitrary measure spaces.

Theorem 4.17. *Let (Ω, Σ, μ) be any finite atomless measure space. Then there exists a sub-σ-algebra $\mathcal{F}$ of Σ, with $\Omega \in \mathcal{F}$, such that the following conditions hold.*

(i) *The conditional expectation operator $M^{\mathcal{F}}$ is a strictly narrow projection on the space $L_1(\mu)$. Consequently, $M^{\mathcal{F}}$ is a strictly narrow projection on any Köthe–Banach space E on (Ω, Σ, μ) whenever $M^{\mathcal{F}}$ is well defined and bounded on E.*

(ii) *There exists a linear isomorphism U from $L_1(\mu)$ onto $L_1(\mathcal{F})$ which sends any function $x \in L_1(\mu)$ to a function $Ux \in L_1(\mathcal{F})$ equimeasurable with x. Hence, for any r.i. Banach space E on (Ω, Σ, μ), $M^{\mathcal{F}}$ is a strictly narrow projection from E onto a subspace isometrically isomorphic to E.*

(iii) *For every r.i. Banach space E on (Ω, Σ, μ) there exists a nonnarrow projection Q of E onto $M^{\mathcal{F}}(E)$. Moreover, if the measure spaces (Ω, Σ, μ) is homogeneous then Q has the following property: there exists a decomposition*

$\Omega = \Omega_1 \oplus \Omega_2$ with $\Omega_i \in \Sigma^+$ such that the restrictions $Q|_{E(\Omega_i)}$ are isomorphic embeddings for $i = 1, 2$.

(iv) *If, moreover, the measure space (Ω, Σ, μ) is homogeneous then the measure spaces $(\Omega, \mathcal{F}, \mu|_{\mathcal{F}})$ and (Ω, Σ, μ) are isomorphic.*

For the proof of Theorem 4.17 we will need the following extension of Theorem 4.10 to a general measure space. As in Section 1.4, we consider $D = \{-1, 1\}$.

Proposition 4.18. *Let N be a countable set and let $N = I \sqcup J$ be a decomposition into infinite subsets. Let $\mathcal{F}_N(I)$ be the minimal σ-algebra of subsets of D^N containing all cylindric sets of the form $\{\xi \in D^N : \xi(i) = \theta_i \text{ for all } i \in F\}$, where F is a finite subset of I and $\theta_i \in D$ for each $i \in F$. Then the conditional expectation operator $M^{\mathcal{F}_N(I)}$ is a strictly narrow projection from $L_1(D^N, \Sigma_N, \mu_N)$ onto $L_1(D^N, \mathcal{F}_N(I), \mu_N|_{\mathcal{F}_N(I)})$.*

Note that $\mathcal{F}_N(I)$ consists of all sets from Σ_N, the characteristic functions of which do not depend on coordinates from J (see Definition 1.13).

Proof. It is enough to construct an isomorphism of the measure space (D^N, Σ_N, μ_N) onto the square $([0,1]^2, \sigma(\Sigma \times \Sigma), \lambda \times \lambda)$, which simultaneously is an isomorphism of $(D^N, \mathcal{F}_N(I), \mu_N|_{\mathcal{F}_N(I)})$ onto $([0,1]^2, \mathcal{G}, (\lambda \times \lambda)|_{\mathcal{G}})$, where the sub-$\sigma$-algebra $\mathcal{G}$ of the Lebesgue σ-algebra $\sigma(\Sigma \times \Sigma)$ on the unit square is defined after the proof of Lemma 4.11, defined in Example 4.9.

By the Carathéodory Theorem 1.16, there exist isomorphisms of the measure spaces $S_I : (D^I, \Sigma_I, \mu_I) \to ([0,1], \Sigma, \lambda)$, and $S_J : (D^J, \Sigma_J, \mu_J) \to ([0,1], \Sigma, \lambda)$. The desired isomorphism of the measure space (D^N, Σ_N, μ_N) onto $([0,1]^2, \sigma(\Sigma \times \Sigma), \lambda \times \lambda)$ is induced by the map $S : D^N \to [0,1]^2$ defined as follows. For any $\xi \in D^N$ let $\xi_I \in D^I$ and $\xi_J \in D^J$ be the restrictions of ξ to I and J, respectively, and put

$$S(\xi) = \big(S_I(\xi_I), S_J(\xi_J)\big) \in [0,1]^2 .$$

Since S_I and S_J are isomorphisms, S is also an isomorphism of the measure spaces. The surjectivity follows from the definition of the measure space on the square $[0,1]^2$. Moreover, $S(\mathcal{F}_N(I)) = \mathcal{G}$ by the construction. □

Proof of Theorem 4.17. **The case when $(\Omega, \Sigma, \mu) = (D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$ for some ordinal α.** We decompose $\omega_\alpha = I_0 \sqcup J_0$ into a disjoint union of subsets with $|I_0| = |J_0| = \aleph_\alpha$. Let $\mathcal{F} = \mathcal{F}_{\omega_\alpha}(I_0)$ be the minimal σ-algebra of subsets of D^{ω_α} containing all cylindric sets of the form $\{\xi \in D^{\omega_\alpha} : \xi(\beta_i) = \theta_i,\ i = 1, \dots, n\}$, where $n \in \mathbb{N}$, $\beta_i \in I_0$ and $\theta_i \in D$ are fixed. We show that $\mathcal{F}$ has the desired properties. Let $\tau : \omega_\alpha \to I_0$ be a bijection. Define a map $S : D^{\omega_\alpha} \to D^{\omega_\alpha}$ by setting for any $\xi \in D^{\omega_\alpha}$

$$\big(S(\xi)\big)(\beta) = \begin{cases} \xi\big(\tau(\beta)\big), & \text{if } \beta \in I_0, \\ 1, & \text{if } \beta \in J_0. \end{cases} \tag{4.2}$$

Proof of (iv). We claim that S induces an isomorphism between the measure spaces $(D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$ and $(D^{\omega_\alpha}, \mathcal{F}, \mu_{\omega_\alpha}|_{\mathcal{F}})$. Indeed, it is enough to notice that S induces a one-to-one measure-preserving correspondence between cylindric sets $A = \{\xi \in D^{\omega_\alpha} : \xi(\beta_i) = \theta_i, 1 \le i \le n\} \in \Sigma_{\omega_\alpha}$ and $S(A) = \{\xi \in D^{\omega_\alpha} : \xi(\tau(\beta_i)) = \theta_i, 1 \le i \le n\} \in \mathcal{F}$, because $\mu_{\omega_\alpha}(A) = 2^{-n} = \mu_{\omega_\alpha}(S(A))$.

Proof of (i). Fix any $A \in \Sigma_{\omega_\alpha}$. Since the set $N_0 \subset \omega_\alpha$ of all coordinates $\beta \in \omega_\alpha$ on which the function $\mathbf{1}_A$ depends, is countable, we can find subsets $I \subseteq I_0$ and $J \subseteq J_0$ such that $|I| = |J| = \aleph_0$ and $N_0 \subseteq I \cup J \stackrel{\text{def}}{=} N$. By Corollary 4.16, the conditional expectation operator $M^{\mathcal{F}_N(I)}$ is a strictly narrow projection on the space $L_1(D^N, \Sigma_N, \mu_N)$.

Now we introduce some more notation. For any $\xi \in D^{\omega_\alpha}$ and a subset $B \subseteq \omega_\alpha$ by ξ_B we denote the restriction of ξ to B. For any $\xi_1 \in D^N$ and $\xi_2 \in D^{\omega_\alpha \setminus N}$ by $\xi_1 \oplus \xi_2$ we denote the element $\xi \in D^{\omega_\alpha}$ such that $\xi_N = \xi_1$ and $\xi_{\omega_\alpha \setminus N} = \xi_2$. For each set $C \in \Sigma_{\omega_\alpha}$ such that the characteristic function $\mathbf{1}_C$ depends only on coordinates $\beta \in N$, we define

$$\varphi(C) = \left\{\xi_1 \in D^N : \mathbf{1}_C(\xi_1 \oplus \xi_2) = 1 \text{ for almost all } \xi_2 \in D^{\omega_\alpha \setminus N}\right\}.$$

Observe that the map $\varphi : \mathcal{F}_{\omega_\alpha}(N) \to \Sigma_N$ defined above is a measure-preserving isomorphism with $\varphi^{-1}(C_1) = \{\xi_1 \in D^{\omega_\alpha} : \mathbf{1}_{C_1}(\xi_N) = 1\}$ for all $C_1 \in \Sigma_N$. Since the operator $M^{\mathcal{F}_N(I)}$ is strictly narrow and φ is a bijection, we can decompose $\varphi(A) = \varphi(A_1) \sqcup \varphi(A_2)$ so that $\varphi(A_i) \in \Sigma_N$ and for $x = \mathbf{1}_{\varphi(A_1)} - \mathbf{1}_{\varphi(A_2)}$ we have $\int_{D^N} x \, d\mu_N = 0$ and $M^{\mathcal{F}_N(I)} x = 0$. Let $y = \mathbf{1}_{A_1} - \mathbf{1}_{A_2}$. Then $\int_{D^{\omega_\alpha}} y \, d\mu_{\omega_\alpha} = 0$ since φ is a measure-preserving map.

We claim that $M^{\mathcal{F}} y = 0$. Indeed, since $A_1, A_2 \in \mathcal{F}_{\omega_\alpha}(N)$, we have that $y = M^{\mathcal{F}_{\omega_\alpha}(N)} y$. Hence, $M^{\mathcal{F}} y = M^{\mathcal{F}_{\omega_\alpha}(I_0)} y = M^{\mathcal{F}_{\omega_\alpha}(I_0)} M^{\mathcal{F}_{\omega_\alpha}(N)} y = M^{\mathcal{F}_{\omega_\alpha}(I)} y$.

Let $B \in F_{\omega_\alpha}(I)$. Then, since the map φ is a measure-preserving isomorphism, we obtain

$$\begin{aligned}
\int_B y \, d\mu_{\omega_\alpha} &= \mu_{\omega_\alpha}(B \cap A_1) - \mu_{\omega_\alpha}(B \cap A_2) \\
&= \mu_N\big(\varphi(B) \cap \varphi(A_1)\big) - \mu_N\big(\varphi(B) \cap \varphi(A_2)\big) \\
&= \int_{\varphi(B)} x \, d\mu_N = 0,
\end{aligned}$$

because $\varphi(B) \in \mathcal{F}_N(I)$ and $M^{\mathcal{F}_N(I)} x = 0$. Thus, by the definition of a conditional expectation operator, $F_{\omega_\alpha}(I) y = 0$.

Proof of (ii). We define a linear isomorphism $U : L_1(D^{\omega_\alpha}) \to L_1(\mathcal{F})$ by setting $(Ux)(t) = x(S^{-1}(t))$ for each $x \in L_1(D^{\omega_\alpha})$ and $t \in D^{\omega_\alpha}$ where S is the measure-preserving isomorphism from $(D^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha})$ to $(D^{\omega_\alpha}, \mathcal{F}, \mu_{\omega_\alpha}|_{\mathcal{F}})$ defined by (4.2). U is surjective since $(U^{-1})(t) = y(s(t))$, $t \in D^{\omega_\alpha}$ is the inverse to the U map. It remains to observe that x and Ux are equimeasurable functions for each

$x \in L_1(D^{\omega_\alpha})$. Indeed, $\mu\big\{t \in D^{\omega_\alpha} : x\big(S^{-1}(t)\big) > a\big\} = \mu\big\{t \in D^{\omega_\alpha} : x(t) > a\big\}$ for each $a \ge 0$, because S is measure preserving.

Proof of (iii). The proof of this item practically does not differ from the proof of Theorem 4.13. We provide it here for the convenience of the reader, due to the difference in notation for the separable case. We set $E_1 = \ker M^{\mathcal{F}}$. Fix any $\beta_0 \in I_0$ and let $C = \{\xi \in D^{\omega_\alpha} : \xi(\beta_0) = 1\}$, $D = \{\xi \in D^{\omega_\alpha} : \xi(\beta_0) = -1\}$, $E_i^C = \{\mathbf{1}_C \cdot x : x \in E_i\}$ and $E_i^D = \{\mathbf{1}_D \cdot x : x \in E_i\}$ for $i = 0, 1$. Observe that $E_i = E_i^C \oplus E_i^D$ for $i = 0, 1$. From (ii) we deduce that $E(C)$ is isometrically isomorphic to E_0^C, which, in turn, is isometrically isomorphic to E_0^D. Then, since E_1^C is a subspace of $E(C)$, there exists an isometric embedding $T^C : E_1^C \to E_0^D$. Analogously, there exists an isometric embedding $T^D : E_1^C \to E_0^C$. Thus the operator $T : E_1 \to E_0$ defined by $T = T^C \oplus T^D$ is an isometric embedding such that $TE_1^C \subseteq E_0^D$ and $TE_1^D \subseteq E_0^C$. By Proposition 4.14, the operator $Q = M^{\mathcal{F}} - T(I - M^{\mathcal{F}})$ is a projection from E onto E_0. We show that Q satisfies the desired properties.

Let $x \in E(C)$. Since $\mathbf{1}_C \cdot M^{\mathcal{F}} x = M^{\mathcal{F}} x$ and $TE_1^C \subseteq E_0^D$, we have that

$$\mathbf{1}_C \cdot Qx = \mathbf{1}_C \cdot M^{\mathcal{F}} x - \mathbf{1}_C \cdot T(I - M^{\mathcal{F}})x = M^{\mathcal{F}} x$$

and

$$\mathbf{1}_D \cdot Qx = \mathbf{1}_D \cdot M^{\mathcal{F}} x - \mathbf{1}_D \cdot T(I - M^{\mathcal{F}})x = T(I - M^{\mathcal{F}})\, x\,.$$

Hence,

$$\begin{aligned}\|Qx\| &\ge \max\Big\{\|\mathbf{1}_C \cdot Qx\|, \|\mathbf{1}_D \cdot Qx\|\Big\} = \max\Big\{\|M^{\mathcal{F}} x\|, \|T(I - M^{\mathcal{F}})x\|\Big\}\\ &= \max\Big\{\|M^{\mathcal{F}} x\|, \|(I - M^{\mathcal{F}})x\|\Big\} \ge \frac{\|M^{\mathcal{F}} x\| + \|(I - M^{\mathcal{F}})x\|}{2} \ge \frac{\|x\|}{2}.\end{aligned}$$

Thus $Q|_{E(C)}$ is an isomorphic embedding. Analogously, $Q|_{E(D)}$ is an isomorphic embedding, and therefore Q is not narrow.

Now we consider the general case. By the Maharam theorem, there exists a decomposition $\Omega = \bigsqcup_{i \in I} \Omega_i$ such that for every $i \in I$ the measure space $(\Omega_i, \Sigma_i, \mu_i)$ is homogeneous, where $\Sigma_i = \{A \cap \Omega_i : A \in \Sigma\}$ and $\mu_i = \mu|_{\Sigma_i}$.

Proof of (i) *and* (ii). For every $i \in I$, using items (i) and (ii) for homogeneous measure spaces, we choose a sub-σ-algebra $\mathcal{F}_i$ of Σ_i so that the conditional expectation operator $M^{\mathcal{F}_i}$ is a strictly narrow projection of the space $L_1(\mu_i)$, and choose a linear isomorphism U_i from $L_1(\mu_i)$ onto $L_1(\mathcal{F}_i)$ that sends any function $x \in L_1(\mu_i)$ to an equimeasurable function $U_i x \in L_1(\mathcal{F}_i)$. Then we set $\mathcal{F} = \{A \in \Sigma : (\forall i \in I)\,(A \cap \Omega_i \in \mathcal{F}_i)\}$, and define a map $U : L_1(\mu) \to L_1(\mathcal{F})$ by setting $Ux = \sum_{i \in I} U_i(x \cdot \mathbf{1}_{\Omega_i})$ for every $x \in L_1(\mu)$. A straightforward verification shows that $M^{\mathcal{F}}$ is a strictly narrow operator on $L_1(\mu)$. The linearity and bijectivity of $U : L_1(\mu) \to L_1(\mathcal{F})$ is obvious. We show that for any $x \in L_1(\mu)$ the functions x and Ux are equimeasurable in modulus.

Indeed, since the sets $(\Omega_i)_{i\in I}$ are disjoint, for any $a > 0$ we have

$$\mu\big\{t \in \Omega : |Ux(t)| < a\big\} = \sum_{i\in I} \mu_i \big\{t \in \Omega_i : |Ux(t)| < a\big\}$$
$$= \sum_{i\in I} \mu_i \big\{t \in \Omega_i : |x(t)| < a\big\} = \mu\big\{t \in \Omega : |Ux(t)| < a\big\}.$$

Proof of (iii). Fix any $i_0 \in I$ and choose a nonnarrow projection Q_{i_0} of $E(\Sigma_{i_0})$ onto $E(\mathcal{F}_i) = M^{\mathcal{F}_i}$. The straightforward verification shows that the operator $Q = M^{\mathcal{F}} - M^{\mathcal{F}_i} + Q_{i_0}$ is a projection from E onto $M^{\mathcal{F}}(E)$. Since $Q|_{E(\Sigma_{i_0})} = Q_{i_0}$, we obtain that Q is not narrow. □

Noncompact narrow operators

As a consequence of Theorem 4.17 we obtain an affirmative answer to Problem 4.5.

Corollary 4.19. *Let E be an r.i. Banach space on a finite atomless measure space (Ω, Σ, μ) such that all the conditional expectation operators are well defined and bounded on E, and let X be a Banach space. If there exists a noncompact operator $T \in \mathcal{L}(E, X)$ then there exists a noncompact narrow operator $S \in \mathcal{L}(E, X)$.*

Proof. Let $\mathcal{F}$ be a sub-σ-algebra of Σ satisfying the conditions of Theorem 4.17. Let $J : M^{\mathcal{F}}(E) \to E$ be a linear isometry. Then the operator $S = T \circ J \circ M^{\mathcal{F}}$ has the desired properties. It is noncompact, because the image of unit ball $S(B_E)$ coincides with $T(B_E)$. By Proposition 1.8, a composition of a narrow operator from the right by a bounded operator is narrow. □

4.3 A characterization of narrow conditional expectation operators

In this section we characterize atomless sub-σ-algebras $\mathcal{F}$ of Σ such that the conditional expectation operator $M^{\mathcal{F}}$ is a narrow (resp., strictly narrow) operator on $L_1(\Omega, \Sigma, \mu)$. For simplicity of the notation, we assume that $\mu(\Omega) = 1$, that is, we consider probability spaces. The results of the section were obtained by Dorogovtsev and the first author in [31].

Narrow and strictly narrow sub-σ-algebras

Definition 4.20. Let (Ω, Σ, μ) be a finite atomless measure space. A sub-σ-algebra $\mathcal{F}$ of Σ is called

- *narrow* if for each $B \in \Sigma$ and each $\varepsilon > 0$ there exists $C \in \Sigma(B)$ such that

$$(\forall\, A \in \mathcal{F}) \qquad \Big|\mu(A \cap C) - \frac{1}{2}\,\mu(A \cap B)\Big| < \varepsilon\,; \tag{4.3}$$

- *strictly narrow* if for each $B \in \Sigma$ there exists $C \in \Sigma(B)$ such that

$$(\forall\, A \in \mathcal{F}) \qquad \mu(A \cap C) = \frac{1}{2}\,\mu(A \cap B)\,. \tag{4.4}$$

Evidently, every strictly narrow sub-σ-algebra is narrow. We will prove that the converse is also true.

Observe that a purely atomic sub-σ-algebra is strictly narrow (cf. Proposition 4.7). Indeed, let $(A_i)_{i\in I}$ be the collection of all atoms of a sub-σ-algebra $\mathcal{F}$ of Σ. For each $B \in \Sigma$ and each $i \in I$ we decompose $A_i \cap B = C_i \sqcup D_i$ into subsets of equal measure $\mu(C_i) = \mu(D_i) = 1/2,\ \mu(A_i \cap B)$. Then $C = \bigcup_{i\in I} C_i \in \Sigma(B)$, and $\mu(A_i \cap C) = \mu(C_i) = 1/2\mu(A_i \cap B)$ for every $i \in I$. Thus $\mathcal{F}$ is strictly narrow.

The following proposition provides a natural family of examples strictly narrow atomless sub-σ-algebras which were the motivation for the term "strictly narrow sub-σ-algebras."

Proposition 4.21. *Let (Ω, Σ, μ) be a finite atomless measure space and $\mathcal{F}$ be a sub-σ-algebra of Σ. Let E a Köthe function space on (Ω, Σ, μ), so that the operator $M^{\mathcal{F}}$ is well defined and bounded on E. Then the operator $M^{\mathcal{F}}$ is strictly narrow if and only if $\mathcal{F}$ is a strictly narrow sub-σ-algebra.*

Proof. Let $\mathcal{F}$ be a strictly narrow sub-σ-algebra and $B \in \Sigma$. We choose $C \in \Sigma(B)$ with property (4.4) and set $x = \mathbf{1}_B - 2\mathbf{1}_C$. Then $x^2 = \mathbf{1}_B$. Since (4.4) holds, in particular, for $A = \Omega$, we have that $\int_\Omega x\,\mathrm{d}\mu = 0$. We claim that $M^{\mathcal{F}} x = 0$. Indeed, for every $A \in \mathcal{F}$ we have

$$\int_A M^{\mathcal{F}} x\,\mathrm{d}\mu = \int_A x\,\mathrm{d}\mu = \int_A \mathbf{1}_B\,\mathrm{d}\mu - 2\int_A \mathbf{1}_C\,\mathrm{d}\mu = \mu(A \cap B) - 2\mu(A \cap C) = 0\,.$$

Thus, $M^{\mathcal{F}}$ is a strictly narrow operator.

Assume that $M^{\mathcal{F}}$ is a strictly narrow operator. Given any $B \in \Sigma$, we choose $x \in \ker M^{\mathcal{F}}$ so that $x^2 = \mathbf{1}_B$. Let $C = \{\omega \in \Omega : x(\omega) = -1\}$. Then $x = \mathbf{1}_B - 2\mathbf{1}_C$. Therefore, for every $A \in \mathcal{F}$ we have

$$\mu(A \cap B) - 2\mu(A \cap C) = \int_A \mathbf{1}_B\,\mathrm{d}\mu - 2\int_A \mathbf{1}_C\,\mathrm{d}\mu = \int_A x\,\mathrm{d}\mu = \int_A M^{\mathcal{F}} x\,\mathrm{d}\mu = 0\,.$$

□

A characterization of strictly narrow sub-σ-algebras

This subsection is devoted to a characterization of strictly narrow sub-σ-algebras in other terms of the measure theory. First we note that the definition of a strictly narrow sub-σ-algebra can be equivalently reformulated as follows.

A sub-σ-algebra $\mathcal{F}$ is strictly narrow provided that for every $B \in \Sigma$ there exists $D \in \Sigma$ of measure $\mu(D) = 1/2$ which is independent of the collection of sets $\mathcal{F}(B) = \{A \cap B : A \in \mathcal{F}\}$.

Indeed, for a given set $B \in \Sigma$ we put $C = B \cap D$. Then the independence means that for any $A \in \mathcal{F}$ we have

$$\mu(A \cap C) = \mu\big(A \cap (B \cap D)\big) = \mu\big((A \cap B) \cap D\big) = \mu(A \cap B)\mu(D) = \frac{1}{2}\mu(A \cap B).$$

Then we apply the technique of conditional measures. Recall that a measure μ on a measurable space (Ω, Σ) is called *perfect*, if for every measurable function $f : \Omega \to \mathbb{R}$ there exists a Borel set $B \subseteq \mathbb{R}$ such that $B \subseteq f(\Omega)$ and $\mu(f^{-1}(B)) = \mu(\Omega)$. The following result is known as the Theorem on the Existence of a Conditional Measure (see [18, Theorems 10.4.5 and 7.5.6]).

Theorem 4.22. *Let (Ω, Σ, μ) be a finite separable measure space with a perfect measure. Then for each sub-σ-algebra $\mathcal{F}$ of Σ there exists a function $p : \Omega \times \Sigma \to \mathbb{R}$, for which the following conditions hold:*

(i) *For every $\omega \in \Omega$ the function $p(\omega, \cdot) : \Sigma \to \mathbb{R}$ is a countably additive measure on Σ, and it is a probability measure if μ is as well.*

(ii) *For any $B \in \Sigma$ the function $p(\cdot, B) : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable and μ-integrable.*

(iii) *For all $A \in \mathcal{F}$ and $B \in \Sigma$ we have $\mu(A \cap B) = \int_A p(\omega, B)\, \mathrm{d}\mu(\omega)$.*

The function $p : \Omega \times \Sigma \to \mathbb{R}$ from Theorem 4.22 is called the *conditional measure* on Σ with respect to $\mathcal{F}$.

Theorem 4.22 implies the following characterization.

Corollary 4.23. *Let (Ω, Σ, μ) be a separable finite measure space with a perfect measure. A sub-σ-algebra $\mathcal{F}$ of Σ is strictly narrow if and only if*

$$(\forall\, B \in \Sigma)(\exists\, C \in \Sigma(B)) \quad p(\omega, C) = \frac{1}{2}\, p(\omega, B) \ \textit{for almost all}\ \ \omega \in \Omega\,, \tag{4.5}$$

where p is the conditional measure on Σ with respect to $\mathcal{F}$.

Proof. Observe that the equality $p(\omega, C) = 1/2\, p(\omega, B)$ for almost all $\omega \in \Omega$ is equivalent to the following equality for each $A \in \mathcal{F}$

$$\int_A p(\omega, C)\, \mathrm{d}\mu(\omega) = \frac{1}{2} \int_A p(\omega, B)\, \mathrm{d}\mu(\omega)\,.$$

Thus, by Theorem 4.22(iii), condition (4.4) holds. □

A canonical example of a strictly narrow sub-σ-algebra is the sub-σ-algebra $\mathcal{G} = \Sigma \times \{[0,1]\}$ of the Lebesgue σ-algebra on the unit square $[0,1]^2$ defined and studied in Section 4.2. Note that in the proof of Theorem 4.10, we actually proved condition (4.5).

Narrow and strictly narrow sub-σ-algebras coincide on complete measure spaces

We now present a characterization of narrow and strictly narrow sub-σ-algebras on complete measure spaces in terms of random variables.

Theorem 4.24. *Let (Ω, Σ, μ) be a separable atomless probability space. Assume that a sub-σ-algebra $\mathcal{F}$ of Σ, and Σ are both complete with respect to the measure μ (that is every subset in $\mathcal{F}$ (resp., Σ) of a set of measure zero belongs to $\mathcal{F}$ (resp., Σ)). Then the following assertions are equivalent:*

(a) *$\mathcal{F}$ is narrow.*

(b) *$\mathcal{F}$ is strictly narrow.*

(c) *There exists a random variable $\xi \in L_0(\mu)$ which is independent of $\mathcal{F}$ and has a nontrivial Gaussian distribution.*

Proof. Observe that the conditional expectation operator $\mathcal{F}$ can be written in terms of conditional measures as follows $(M^{\mathcal{F}}\varphi)(\omega) = \int_\Omega \varphi(\omega')\, p\,(\omega, d\omega')$.

Proof of (a) $\Rightarrow$ (b). Let $B \in \Sigma^+$. Since $\mathcal{F}$, for each $n \geq 1$ there exists $C_n \in \Sigma(B)$ such that for every $A \in \mathcal{F}$ we have

$$\left|\mu(A \cap C_n) - \frac{1}{2}\,\mu(A \cap B)\right| < \frac{1}{n}\,. \tag{4.6}$$

We rewrite (4.6), using conditional measures p, i.e. for all $A \in \mathcal{F}$ we have

$$\left|\int_A p\,(\omega, C_n)\,\mu(d\omega) - \frac{1}{2}\int_A p(\omega, B)\,\mu(d\omega)\right| < \frac{1}{n}\,. \tag{4.7}$$

We claim that the sequence $p(\cdot, C_n)$ converges in measure to $\frac{1}{2}p(\cdot, B)$. Indeed, otherwise there would exist $\delta_1, \delta_2 > 0$ so that (passing if necessary to a subsequence) for all $n \geq 1$, $\mu\{\omega : |p\,(\omega, C_n) - \frac{1}{2}\,p\,(\omega, B)| > \delta_1\} > \delta_2$. Without loss of generality, we may and do assume that for all $n \geq 1$,

$$\mu\Big\{\omega :\ p\,(\omega, C_n) - \frac{1}{2}\,p\,(\omega, B) > \delta_1\Big\} > \frac{\delta_2}{2}\,.$$

Define $A_n = \{\omega :\ p(\omega, C_n) - \frac{1}{2}\,p(\omega, B) > \delta_1\}$. Then $\mu(A_n) > \frac{\delta_2}{2}$, and, by Theorem 4.22(iii) and the definition of A_n, we get

$$\begin{aligned}\mu(A_n \cap C_n) &= \int_{A_n} p\,(\omega, C_n)\,\mu(d\omega) > \int_{A_n}\Big(\frac{1}{2}\,p\,(\omega, B) + \delta_1\Big)\mu(d\omega)\\ &> \frac{1}{2}\int_{A_n} p\,(\omega, B)\,\mu(d\omega) + \frac{\delta_1\delta_2}{2} = \mu(A_n \cap B) + \frac{\delta_1\delta_2}{2}.\end{aligned} \tag{4.8}$$

Inequality (4.8) contradicts (4.6), and the claim is proved.

So, without loss of generality, we assume that

$$p(\cdot, C_n) \to \frac{1}{2} p(\cdot, B), \; n \to \infty \text{ in measure} \tag{4.9}$$

and the convergence (4.9) holds on the set $\Omega_1 \in \mathcal{F}$ of full measure, i.e. $\mu(\Omega \setminus \Omega_1) = 0$. Consider the measurable space $(B, \Sigma(B))$ with the measure $p(\omega, \cdot)$ for a fixed ω and choose $C \in \Sigma(B)$ so that

$$p(\omega, C) = \frac{1}{2} p(\omega, B) . \tag{4.10}$$

A priori C depends on ω. However, by [18, p. 454], the set Ω is decomposed into equivalence classes which are atoms of the sub-σ-algebra $\mathcal{F}$ so that for each $\omega_1, \omega_2 \in \Omega$ from the same atom, the measures $p(\omega_1, \cdot)$ and $p(\omega_2, \cdot)$ coincide and are supported on that atom. Thus, condition (4.9) is equivalent to the following one:

$$\forall\, \omega \in \Omega_B \qquad p(\omega, C_n^\omega) \to \frac{1}{2} p(\omega, B^\omega), \; n \to \infty . \tag{4.11}$$

Here Ω_B is a $\mathcal{F}$-measurable set of measure one, C_n^ω and B^ω denote the intersections of C_n and B, respectively, with the atom of the sub-σ-algebra $\mathcal{F}$, which contains ω.

Thus for every $\omega \in \Omega_1$ there exists C^ω such that $p(\omega, C^\omega) = \frac{1}{2} p(\omega, B^\omega)$ and $C = \bigcup_{\omega \in \Omega_1} C^\omega \in \Sigma$. Hence for each $A \in \mathcal{F}$ we have

$$\mu(A \cap C) = \int_{A \cap \Omega_1} p(\omega, C^\omega)\, \mu(d\omega) = \frac{1}{2} \int_{A \cap \Omega_1} p(\omega, B^\omega)\, \mu(d\omega) = \frac{1}{2} \mu(A \cap B).$$

Thus (a) $\Rightarrow$ (b) is proved.

Proof of (b) $\Rightarrow$ (c). We consider the conditional measures $p(\cdot, \cdot)$. Since $\mathcal{F}$ is strictly narrow, for every $B \in \Sigma^+$ there exists $C \in \Sigma$, such that for some subset Ω_B of measure one the following equality holds

$$\forall\, \omega \in \Omega_B \qquad p(\omega, C^\omega) = \frac{1}{2} p(\omega, B^\omega) . \tag{4.12}$$

Note that condition (4.12) holds if and only if the sub-σ-algebra $\mathcal{F}$ is strictly narrow.

We construct a sequence of decompositions of Ω. Set $A_{0,0} = \Omega$ and choose $A_{1,0}$ so that (4.12) holds with $B = A_{0,0}$ and $C = A_{1,0}$. Let $A_{1,1} = A_{0,0} \setminus A_{1,0}$. Then, taking as B the set $A_{1,0}$, we construct $A_{2,0}$ and $A_{2,1}$. Next, using the set $A_{1,1}$ we construct $A_{2,2}$ and $A_{2,3}$. Continuing the construction, we obtain a sequence $\{A_{n,k}; n \geq 0, k = 0, \ldots, 2^{n-1}\}$ of decompositions of the set Ω. Then there exists a set Ω_1 of measure one such that for every $\omega \in \Omega_1$ and every $n \geq 0, k = 0, \ldots, 2^{n-1}$ we have (recall that for a set A, the set A^ω is the intersection of A with the atom of $\mathcal{F}$, which contains ω)

$$p(\omega, A_{n,k}^\omega) = \frac{1}{2^n} , \tag{4.13}$$

and $A^{\omega}_{n+1,2k} \cup A^{\omega}_{n+1,2k+1} = A^{\omega}_{n,k}$. Fix any number $t \in (0;1)$ and consider its dyadic expansion $t = \sum_{n=1}^{\infty} 2^{-n}\varepsilon_n$, with $\varepsilon_n \in \{0,1\}$. Define a set $\Delta_t \in \Sigma$ as follows:

$$\Delta_t = \bigcup_{n=1}^{\infty} (A_{n,k_n})_{\varepsilon_n} , \tag{4.14}$$

$$\text{where } (A_{n,k_n})_{\varepsilon_n} = \begin{cases} \emptyset, & \text{if } \varepsilon_n = 0, \\ A_{n,k_n}, & \text{if } \varepsilon_n = 1, \end{cases}$$

and $k_n = \min\{k \in \mathbb{N} : A_{n,k} \not\subseteq \bigcup_{m=1}^{n-1} (A_{m,k})_{\varepsilon_m}\}$. By (4.13), $\mu(\Delta_t) = t$, and Δ_t is independent of $\mathcal{F}$. Moreover, $\Delta_{t_1} \subseteq \Delta_{t_2}$ for $t_1 < t_2$ by the construction. Then one can construct a Gaussian random variable which is independent of $\mathcal{F}$ using the sets $\Delta_t, t \in (0;1)$ in the usual way.

Proof of (c) $\Rightarrow$ (b). Let ξ be a Gaussian random variable with a nontrivial distribution and which is independent of $\mathcal{F}$. Consider sets $\Delta_t = \{\xi < a_t\}$ such that $\mu(\Delta_t) = t$ for all $t \in (0,1)$. Then we construct decompositions $\{A_{n,k}\}$ with property (4.14). For each $B \in \Sigma$ and each n,k the function $\omega \mapsto p(\omega, A^{\omega}_{n,k} \cap B^{\omega})$ is $\mathcal{F}$-measurable and

$$p\Big(\omega, \bigcup_{n=1}^{N} A^{\omega}_{n,k_n} \cap B^{\omega}\Big) \nearrow \frac{1}{2}\, p(\omega, B^{\omega}), \quad \text{as } N \to \infty .$$

Hence, we can construct a set C as follows $C = \bigcup_{\omega\in\Delta_1} C^{\omega}$, where $C^{\omega} = \Big(\bigcup A^{\omega}_{n,k_n}\Big) \cap B^{\omega}$. Then with this C condition (4.10) is satisfied, and thus $\mathcal{F}$ is strictly narrow. □

Combining Proposition 4.21 with Theorem 4.24, we obtain the following result.

Corollary 4.25. *Let (Ω, Σ, μ) be a separable atomless probability space. Assume that a sub-σ-algebra $\mathcal{F}$ of Σ as well as Σ itself are complete with respect to the measure μ. Let E be a Köthe–Banach space on (Ω, Σ, μ), such that the conditional expectation operator $M^{\mathcal{F}}$ is well defined and bounded. Then the following conditions are equivalent:*

(a) *$M^{\mathcal{F}}$ is narrow.*

(b) *$M^{\mathcal{F}}$ is strictly narrow.*

(c) *There exists a random variable ξ on (Ω, Σ, μ) which is independent of $\mathcal{F}$ and has a nontrivial Gaussian distribution.*

Chapter 5

Ideal properties, conjugates, spectrum and numerical radii of narrow operators

In Section 1.3 we proved that the set of all narrow operators has the left-ideal property (Proposition 1.8). However, in general, it has no other ideal-type properties. In Section 5.1 we show that the right-ideal property fails in most spaces and that in most "good" spaces, every operator is a sum of two narrow operators (however, a sum of two narrow operators on L_1 is narrow, see Theorem 7.46 below). On the other hand, in r.i. spaces on $[0, 1]$, other than L_∞, the sum of a narrow and a compact operator is narrow (Proposition 5.5). In Section 5.2 we show that, in contrast to compact operators, the conjugate operator to a narrow operator need not be narrow. In Section 5.3 we present a result of Krasikova [70] that every compact subset of $\mathbb{C} \setminus \{0\}$ can be a spectrum for some narrow operator. In Section 5.4 we study whether the numerical index of L_p for $1 < p < \infty$, $p \neq 2$, can be approximated by numerical radii of narrow operators on L_p. Our technique shows that the behavior of any narrow operator on $L_p(\mu)$ is very close, in a certain sense, to that of a rank-one operator (see Lemma 5.17).

5.1 Ideal properties of narrow operators and stability of rich subspaces

Proposition 1.8 shows that the set of all narrow operators has the left-ideal property. However, in most cases, the right-ideal property does not hold.

Proposition 5.1. *Let E be an r.i. Banach space on a finite atomless measure space (Ω, Σ, μ). Then there exist operators $T, S \in \mathcal{L}(E)$ with S narrow and ST nonnarrow.*

Proof. By Theorem 4.17 we choose a complemented subspace E_1 of E isomorphic to E, an isomorphism $T : E \to E_1$ and a narrow projection S of E onto E_1. Then $ST = T$ is not narrow. □

Every operator on a "good" space is a sum of two narrow operators

Is the sum of two narrow operators narrow? It is a striking and very interesting phenomenon that if an r.i. Banach space on $[0, 1]$ has an unconditional basis then the answer is negative, while for $E = L_1$ it is affirmative. No less interesting is that the first fact is quite simple, and the second is quite complicated (see Section 7.2).

We start by showing a simple argument that every operator on L_p, $1 < p < \infty$, is a sum of two narrow operators. In fact, even more is true.

Theorem 5.2. *Let E be an r.i. Banach space on* $[0,1]$ *with an unconditional basis. Then the identity I of E is a sum $I = P + Q$ of two narrow projections $P, Q \in \mathcal{L}(E)$. Consequently, every operator $T \in \mathcal{L}(E)$ is a sum $T = T_1 + T_2$ of two narrow operators $T_1, T_2 \in \mathcal{L}(E)$.*

Proof. Let $h_{0,0}, (h_{m,i})_{m=0}^{\infty}{}_{i=1}^{2^m}$ be the L_∞-normalized Haar system. Decompose the set of integers into two infinite parts $\mathbb{N} = N_0 \sqcup N_1$ and let

$$E_0 = \overline{\operatorname{span}}\big(h_{0,0},\, h_{m,i} : m \in N_0,\, i \in \{1, \dots 2^m\}\big),$$
$$E_1 = \overline{\operatorname{span}}\big(h_{0,1},\, h_{m,i} : m \in N_1,\, i \in \{1, \dots 2^m\}\big).$$

Since the Haar system is unconditional in E [80, p. 156], we have $E = E_0 \oplus E_1$. Now we show that both corresponding projections P_0 onto E_0 and P_1 onto E_1 are narrow. Let $I_n^k = [2^{-n}(k-1), 2^{-n}k)$ be any dyadic interval, and let $\varepsilon > 0$. Fix any index $j \in \{0,1\}$ and choose any $m \in N_{1-j}$ with $m > n$. Since $2^{-n}(k-1) = 2^{-m}(k-1)2^{m-n}$, we have that $I_n^k = \bigsqcup_{i=(k-1)2^{m-n}}^{k2^{m-n}-1} I_m^i = \bigsqcup_{i=(k-1)2^{m-n}}^{k2^{m-n}-1} \operatorname{supp} h_{m,i}$.

Set

$$x = \sum_{i=(k-1)2^{m-n}}^{k2^{m-n}-1} h_{m,i}$$

and observe that $x^2 = \mathbf{1}_{I_n^k}$. Since $x \in E_{1-j} = \ker P_j$, we have that $\|P_j x\| = 0 < \varepsilon$.

Since E is a separable r.i. space, it has an absolute continuous norm on the unit. Indeed, otherwise there is $\delta > 0$ such that $\|\mathbf{1}_A\| \ge \delta$ for every $A \in \Sigma^+$, and hence, the set $(\mathbf{1}_{[0,t]})_{t \in [0,1]}$ is δ-separated and uncountable, which contradicts separability of E. Therefore, by Lemma 1.12, P_j is narrow. □

Corollary 5.3. *Let E be an r.i. Banach space on* $[0,1]$ *with an unconditional basis. Then E is a direct sum of two rich subspaces $E = E_0 \oplus E_1$.*

As one can see from the proof of Theorem 5.2, subspaces E_0 and E_1 are rich in any Köthe–Banach space E on $[0,1]$ with absolutely continuous norm on the unit. If, in addition, the Haar system is a basis of E then the subspaces E_0 and E_1 are *quasi-complemented* in E, that is, $E_0 \cap E_1 = \{0\}$ and $\overline{\operatorname{span}}(E_0 \cup E_1) = E$ (this property follows directly from the definition of a basis).

Since the Haar system is a basis of every separable r.i. space [80, p. 150], we obtain the following statement.

Proposition 5.4. *Let E be a Köthe–Banach space on* $[0,1]$ *such that the Haar system is a basis of E (in particular, a separable r.i. Banach space). Then E has a pair of rich quasi-complemented subspaces.*

The sum of a narrow and a compact operator is narrow

Since a sum of two narrow operators does not have to be narrow, it is natural to ask whether a sum of a narrow and other "small" operator, such as a compact operator, is narrow. The answer is affirmative in a large class of spaces.

Proposition 5.5. *Let E be an r.i. Banach space on $[0,1]$, not equal to L_∞, up to an equivalent norm. Let $T \in \mathcal{L}(E,X)$ be narrow and let $K \in \mathcal{L}(E,X)$ be compact. Then $T + K$ is narrow.*

Proof. Let $A \in \Sigma$. Using the definition of a narrow operator, we construct a Rademacher system (r_n) on A such that $\lim_{n\to\infty} \|Tr_n\| = 0$. By Rodin–Semenov's result [121] (see also [80, p. 160]), (r_n) is weakly null. Thus, $\lim_{n\to\infty} \|Kr_n\| = 0$. Hence, $\lim_{n\to\infty} \|(T+K)\,r_n\| = 0$. □

As we will see later, a compact operator on L_∞ need not be narrow, see Section 11.4. Thus, Proposition 5.5 is false for the setting of operators on L_∞. However, we do not know the following.

Open problem 5.6. Is a sum of two narrow operators from $\mathcal{L}(L_\infty)$, at least one of which is compact, narrow?

We will show in Section 7.2 that the set of all narrow operators on L_1 is a band in $\mathcal{L}(L_1)$. This is the strongest known positive result about algebraic properties of the set of narrow operators.

Stability of rich subspaces

The notion of a rich subspace is stable in the sense of the following propositions. First observe that if $K \in \mathcal{L}(X)$ is a compact operators on a Banach space X then the image $(I+K)(X)$ is closed in X.

Proposition 5.7. *Let E be an r.i. Banach space on $[0,1]$, not equal to L_∞, up to an equivalent norm. Let $T \in \mathcal{L}(E)$ be a compact operator. If X is a rich subspace of E then so is $(I+K)(X)$.*

Proof. Let $A \in \Sigma^+$. Using the definition of a rich subspace, we choose a sequence $x_n \in X$ and a Rademacher system (r_n) on A such that $\|x_n - r_n\|$ and $\|x_n - r_n\| \to 0$. By Rodin–Semenov's result [121], (r_n) is weakly null. Since K is compact, $\|y_n - r_n\| = \|Kr_n\| \to 0$, where $y_n = x_n + Kr_n \in (I+K)(X)$. Thus, $(I+K)(X)$ is rich. □

Corollary 5.8. *Let E be an r.i. Banach space on $[0,1]$, not equal to L_∞, up to an equivalent norm. If X is a rich subspace of E and Y is a subspace of X with* $\dim X/Y < \infty$ *then Y is also rich.*

Proposition 5.9. *Let E be an r.i. Banach space on $[0,1]$, not equal to L_∞, up to an equivalent norm, and let X be a rich subspace of E with a normalized basis (x_n). If a sequence (y_n) in E satisfies $\sum_{n=1}^\infty \|x_n - y_n\| < \infty$ then the subspace $Y = [y_n]$ is rich.*

Proof. Let K be the basis constant of (x_n). Fix any $A \in \Sigma^+$ and $\varepsilon \in (0,2)$, and choose $n_0 \in \mathbb{N}$ so that $\sum_{n=n_0}^\infty \|x_n - y_n\| < \varepsilon/(4K)$. By [79, p. 5], there exists a continuous linear operator $T : [x_n]_{n=n_0}^\infty \to [y_n]_{n=n_0}^\infty$ with $Tx_n = y_n$ for each $n \geq n_0$ and $\|Tx - x\| \leq \varepsilon\|x\|/4$ for each $x \in [x_n]$. By Corollary 5.8, $[x_n]_{n=n_0}^\infty$ is a rich subspace of E. Now choose $x \in [x_n]_{n=n_0}^\infty$ and $y \in E$ so that $y^2 = \mathbf{1}_A$, $\int_{[0,1]} y \, d\mu = 0$ and $\|x - y\| < \varepsilon/2$. Then $Tx \in Y$, $\|x\| \leq \|y\| + \varepsilon/2 \leq 1 + \varepsilon/2 < 2$ and $\|Tx - y\| \leq \|Tx - x\| + \|x - y\| \leq \varepsilon\|x\|/4 + \varepsilon/2 < \varepsilon$. □

5.2 Conjugates of narrow operators need not be narrow

As a consequence of Theorem 4.17 we obtain that the conjugate operator to a narrow operator need not be narrow, in contrast to compact operators.

Corollary 5.10. *Let (Ω, Σ, μ) be a finite atomless measure space and $1 < p < \infty$. There exists a narrow operator $T \in \mathcal{L}(L_p(\mu))$ such that $T^* \in \mathcal{L}(L_q(\mu))$ is not narrow, where $1/p + 1/q = 1$.*

Proof. Let $\mathcal{F}$ be a sub-σ-algebra of Σ satisfying the conditions of Theorem 4.17. Let $J : M^{\mathcal{F}}(L_p(\mu)) \to L_p(\mu)$ be an onto isomorphism. Then the operator $S = J \circ M^{\mathcal{F}}$ is narrow and onto. However, the conjugate operator S^* is an into isomorphism and thus nonnarrow. □

Note that Corollary 5.10 also holds for r.i. Banach spaces E on a finite atomless measure space (Ω, Σ, μ) such that E^* can be represented as an r.i. Banach space on (Ω, Σ, μ), and such that all the conditional expectation operators are well defined and bounded on both E and E^*.

5.3 Spectrum of a narrow operator

It is well known that the spectrum of a compact operator is a countable subset of $\mathbb{C}$ having, at most, one limiting point zero. In contrast, the only restriction on a compact subset $K \subset \mathbb{C}$ to be a spectrum of a narrow operator is that $0 \in K$.

Theorem 5.11. *([70]) Let E be a complex Köthe–Banach space with an absolutely continuous norm on the unit on a finite atomless measure space (Ω, Σ, μ) such that there exists an infinite purely atomic sub-σ-algebra $\mathcal{F}$ of Σ such that the conditional expectation operator $M^{\mathcal{F}}$ is well defined and bounded in E. Then a subset $K \subset \mathbb{C}$*

is a spectrum of some narrow operator $T \in \mathcal{L}(E)$ if and only if K is a compact set containing zero.

Proof. The "only if" part is obvious. To prove the "if," assume $0 \in K \subset \mathbb{C}$ and K is a compact set. We choose a sequence $(k_n)_{n=1}^{\infty}$ dense in K (as $K \neq \emptyset$, it is possible). Let $\mathcal{F}$ be generated by a disjoint sequence $(\Omega_n)_{n=1}^{\infty}$ with $\Omega_n \in \Sigma^+$. Then $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$. Given $x \in E$, we set

$$(Tx)(\omega) = \frac{k_n}{\mu(\Omega_n)} \int_{\Omega_n} x \, \mathrm{d}\mu \tag{5.1}$$

for each $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Observe that

$$(M^{\mathcal{F}} x)(\omega) = \frac{1}{\mu(\Omega_n)} \int_{\Omega_n} x \, \mathrm{d}\mu$$

for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Since $C = \sup l\{|k| : k \in K\} < \infty$, we have that $|(Tx)(\omega)| \le C(M^{\mathcal{F}}|x|)(\omega)$ for all $\omega \in \Omega$, hence, $Tx \in E$ and $\|Tx\| \le C\,\|M\|\|x\|$ for all $x \in E$. Thus, (5.1) defines a bounded linear operator on E.

Now we show that T is narrow. Let $A \in \Sigma$. For every $n \in \mathbb{N}$, we set $A_n = A \cap \Omega_n$. Since the restriction of T to $E(\Omega_n)$ is a rank-one operator, this restriction is narrow. Therefore, for every $n = 1, 2, \ldots$ there is $x_n \in E(\Omega_n)$ such that $x_n^2 = \mathbf{1}_{A_n}$, $\int_{\Omega_n} x_n \, \mathrm{d}\mu = 0$ and $\|Tx_n\| < 2^{-n}\varepsilon$. Now we define $x(\omega) = x_n(\omega)$ for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then $x^2 = \mathbf{1}_A$, $x \in E$, $\int_{\Omega} x \, \mathrm{d}\mu = 0$ and $\|Tx\| \le \sum_{n=1}^{\infty} \|Tx_n\| < \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon$. Thus, T is narrow.

It remains to find the spectrum $\sigma(T)$ of T. Since the numbers k_n for each $n = 1, 2, \ldots$ are eigenvalues of T, we have that all of them belong to $\sigma(T)$. And since $\sigma(T)$ is closed, $K = \overline{\{k_n : n \in \mathbb{N}\}} \subseteq \sigma(T)$. Now we show the converse inclusion, that is, if $\lambda \in \mathbb{C} \setminus K$ then the operator $T - \lambda I$ is an isomorphism. Given $y \in E$, we show that the equation $(T - \lambda I)x = y$ has a unique solution in E. Integrating the equation over Ω_n, we obtain

$$\int_{\Omega_n} x \, \mathrm{d}\mu = (k_n - \lambda)^{-1} \int_{\Omega_n} y \, \mathrm{d}\mu\,.$$

Hence, the initial equation is equivalent to

$$\frac{k_n}{\mu(\Omega_n)}(k_n - \lambda)^{-1} \int_{\Omega_n} y \, \mathrm{d}\mu - \lambda x(\omega) = y(\omega)$$

for almost all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, which obviously has the solution

$$x(\omega) = \frac{k_n}{\lambda(k_n - \lambda)\mu(\Omega_n)} \int_{\Omega_n} y \, \mathrm{d}\mu - y(\omega)$$

for almost all $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then

$$\begin{aligned}\left\|(T - \lambda I)^{-1} y\right\| &\le \left(\frac{|k_n| \, \|M^{\mathcal{F}}\|}{|\lambda||k_n - \lambda|} + 1\right) \|y\| \\ &\le \left(\frac{C \, \|M^{\mathcal{F}}\|}{d_\lambda^2} + 1\right) \|y\|,\end{aligned}$$

where $d_\lambda = \inf\{|\lambda - \alpha| : \alpha \in K\} > 0$, thus, $(T - \lambda I)^{-1} \in \mathcal{L}(E)$. □

5.4 Numerical radii of narrow operators on $L_p(\mu)$-spaces

The numerical index of a Banach space X is the constant depending on numerical ranges of operators on X. The notion of numerical range was first introduced by Toeplitz in 1918 for matrices. The same definition was used for operators on arbitrary Hilbert spaces, and in the sixties it was extended to operators on general Banach spaces by Lumer and Bauer.

The *numerical range* $V(T)$ of an operator $T \in \mathcal{L}(X)$ is defined by

$$V(T) = \{x^*(Tx) : x \in X, x^* \in X^*, \|x\| = \|x^*\| = 1, x^*(x) = 1\},$$

and the *numerical radius* is the seminorm defined on $\mathcal{L}(X)$ by

$$v(T) := \sup\{|\lambda| : \lambda \in V(T)\}.$$

The numerical range of a bounded linear operator is connected, but not necessarily convex, and, in the complex case, its closure contains the spectrum of the operator.

The *numerical index* of a Banach space X is the constant given by

$$n(X) = \inf\{v(T) \ : \ T \in \mathcal{L}(X), \ \|T\| = 1\}.$$

Evidently, $0 \le n(X) \le 1$ for every Banach space X, and $n(X) > 0$ means that the numerical radius and the operator norm are equivalent on $\mathcal{L}(X)$. In the real case, all values in [0, 1] are possible for the numerical index. In the complex case we have $1/e \le n(X) \le 1$ and all of these values are possible. It is known that $n(X) \le n(X^*)$, and $n(X) = n(X^*)$ if X is reflexive. There are some classical Banach spaces for which the numerical index has been calculated. For instance, the numerical index of $L_1(\mu)$ is 1. If H is a Hilbert space with $\dim X > 1$ then $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case.

An important still open problem in the theory is the question of evaluating the numerical index $n(L_p)$ for $1 < p < \infty$, $p \ne 2$ [35, p. 488]. It is known that $n(L_p(\mu)) > 0$ for $p \ne 2$ [89], and that $n(L_p(\mu)) = \inf_m n(\ell_p^m)$ for every measure μ if $\dim(L_p(\mu)) = \infty$ [91]. In particular, $n(L_p) = n(\ell_p)$. For other interesting facts and problems on the numerical index see [55].

It is very natural to ask whether the numerical index $n(L_p)$ can be approximated by numerical radii of finite rank operators, or compact operators, or even narrow operators. This is true in ℓ_p, but for L_p these questions remain open. However, in [90] Martín, Merí and Popov obtained some partial results. Here we discuss the following problem posed in [90].

Open problem 5.12. Let $1 < p < \infty$, $p \neq 2$.

$$n_{\text{nar}}(L_p) = \inf\{v(T) \ : \ T \in \mathcal{L}(L_p),\ \|T\| = 1,\ T \text{ is narrow}\}.$$

Does $n(L_p) = n_{\text{nar}}(L_p)$?

For ℓ_p, $n(\ell_p)$ coincides with the numerical index of compact operators on ℓ_p, and even with the numerical index of finite rank operators.

Proposition 5.13. *Let* $1 < p < \infty$. *Then*

$$n(\ell_p) = \inf\{v(T) \ : \ T \in \mathcal{L}(\ell_p),\ \|T\| = 1,\ \text{rank}(T) < \infty\}.$$

Proof. Let $T \in \mathcal{L}(\ell_p)$, $\|T\| = 1$ and $\varepsilon > 0$. We show that there is $S \in \mathcal{L}(\ell_p)$ with $\text{rank}(S) < \infty$, $\|S\| = 1$ and $v(S) \le (1+\varepsilon)v(T)$. For each $n \in \mathbb{N}$, let P_n be the natural projection from ℓ_p onto the span of $\{e_i\}_{i=1}^n$. Let $T_n = P_n T P_n$. We claim that $v(T_n) \le v(T)$ for each n.

Indeed, fix n and $x \in \ell_p$ with $\|x\| = 1$. Note that there exists a unique element $x^\# \in \ell_q$, $q = p/(p-1)$, so that $\|x^\#\| = \|x\| = 1$ and $\langle x^\#, x\rangle = 1$. This unique element is given by

$$x^\# = \begin{cases} |x|^{p-1}\,\text{sign}(x) & \text{in the real case,} \\ |x|^{p-1}\,\text{sign}(\overline{x}) & \text{in the complex case.} \end{cases}$$

Note that

$$(P_n x)^\# = P_n^*(x^\#). \tag{5.2}$$

If $P_n x = 0$ then $|\langle x^\#, T_n x\rangle| = 0 \le v(T)$. If $P_n x \neq 0$, then let $y = \|P_n x\|^{-1} P_n x$ and observe that by (5.2)

$$y^\# = \frac{(P_n x)^\#}{\|P_n x\|^{p-1}} = \frac{P_n^*(x^\#)}{\|P_n x\|^{p-1}}.$$

Hence,

$$\left|\langle x^\#, T_n x\rangle\right| = \left|\langle P_n^* x^\#, T(P_n x)\rangle\right| = \|P_n x\|\left|\langle y^\#, T y\rangle\right| \le v(T).$$

By arbitrariness of x with $\|x\| = 1$, $v(T_n) \le v(T)$.

Let n so that $\|T_n\| \ge (1+\varepsilon)^{-1}$ and set $S = \|T_n\|^{-1} T_n$. Then $\|S\| = 1$, $\text{rank}(S) < \infty$ and $v(S) = \|T_n\|^{-1} v(T_n) \le (1+\varepsilon)v(T)$. □

The above proof cannot be extended to L_p, since (5.2) no longer holds if we consider the basic projections associated to the Haar system, or even if (P_n) are the conditional expectation operators with respect to the sub-σ-algebra generated by the dyadic intervals of length 2^{-n}.

The remainder of this section is devoted to the proof of lower estimates of $n_{\mathrm{nar}}(L_p)$ in the complex and real cases.

We define the following:

$$\kappa_p = \max_{\tau>0} \frac{\tau^{p-1}}{1+\tau^p} = \max_{\lambda\in[0,1]} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \frac{1}{p^{1/p}q^{1/q}} . \tag{5.3}$$

The number κ_p is the numerical radius of the operator $T(x,y) = (y,0)$ defined on the real or complex space ℓ_p^2, see [88, Lemma 2] for instance.

Theorem 5.14. *([90]) Let (Ω,Σ,μ) be an atomless finite measure space. Then, for every $1 < p < \infty$ we have*

$$n_{\mathrm{nar}}(L_p(\mu)) \geq \kappa_p^2 \ \textit{ in the complex case },$$

$$n_{\mathrm{nar}}(L_p(\mu)) \geq \max_{\tau>0} \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p} \ \textit{ in the real case }.$$

Notice that the inequality for the real case gives a positive estimate for $1 < p < \infty$ $(p \neq 2)$ which tends to 1 as $p \to 1$ or $p \to \infty$.

To prove this result we need the following lemmas which suggest that a narrow operator behaves almost like a rank-one operator when it is restricted to a suitable finite dimensional subspace of arbitrarily large dimension.

Lemma 5.15. *Let (Ω,Σ,μ) be an atomless finite measure space, $1 \leq p < \infty$, $T \in \mathcal{L}(L_p(\mu))$ a narrow operator, $x \in L_p(\mu)$ a simple function, $Tx = y$, $\varepsilon > 0$ and $\Omega = D_1 \sqcup \ldots \sqcup D_\ell$ any partition. Then there exists a partition $\Omega = A \sqcup B$ so that*

(i) $\|x_A\|^p = \|x_B\|^p = 2^{-1}\|x\|^p$;

(ii) $\mu(D_j \cap A) = \mu(D_j \cap B) = \frac{1}{2}\mu(D_j)$ *for each* $j = 1,\ldots,\ell$;

(iii) $\|Tx_A - 2^{-1}y\| < \varepsilon$ *and* $\|Tx_B - 2^{-1}y\| < \varepsilon$.

Proof. Let $x = \sum_{k=1}^m a_k \mathbf{1}_{C_k}$ for some $a_k \in \mathbb{K}$ and $\Omega = C_1 \sqcup \ldots \sqcup C_m$. For each $k = 1,\ldots,m$ and $j = 1,\ldots,\ell$ define sets $E_{k,j} = C_k \cap D_j$ and, using the definition of narrow operator, choose $u_{k,j} \in L_p(\mu)$ so that

$$u_{k,j}^2 = \mathbf{1}_{E_{k,j}}, \qquad \int_\Omega u_{k,j}\, d\mu = 0, \qquad \text{and} \qquad |a_k|\|Tu_{k,j}\| < \frac{2\varepsilon}{m\ell} .$$

Let

$$E_{k,j}^+ = \{t \in E_{k,j} \ : \ u_{k,j}(t) \geq 0\}, \qquad E_{k,j}^- = E_{k,j} \setminus E_{k,j}^+$$

which satisfy $\mu(E^+_{k,j}) = \mu(E^-_{k,j}) = \frac{1}{2}\mu(E_{k,j})$, and define

$$A = \bigcup_{k=1}^{m} \bigcup_{j=1}^{\ell} E^+_{k,j} \qquad \text{and} \qquad B = \bigcup_{k=1}^{m} \bigcup_{j=1}^{\ell} E^-_{k,j} \,.$$

We will show that the partition $\Omega = A \sqcup B$ has the desired properties. Indeed,

$$\begin{aligned}\|x_A\|^p &= \sum_{k=1}^{m}\sum_{j=1}^{\ell} |a_k|^p \mu(E^+_{k,j}) = \sum_{k=1}^{m} |a_k|^p \sum_{j=1}^{\ell} \frac{\mu(E_{k,j})}{2} = \sum_{k=1}^{m} |a_k|^p \frac{\mu(C_k)}{2} \\ &= \frac{\|x\|^p}{2}\end{aligned}$$

and, obviously, $\|x_B\|^p = \|x_A\|^p$. Thus (i) is proved.

Since $E^+_{k,j} \subseteq E_{k,j} \subseteq D_j$, for each $j_0 \in \{1,\dots,\ell\}$, we have that

$$D_{j_0} \cap A = \bigcup_{k=1}^{m} \bigcup_{j=1}^{\ell} \left(D_{j_0} \cap E^+_{k,j}\right) = \bigcup_{k=1}^{m} E^+_{k,j_0}$$

and hence

$$\mu\big(D_{j_0} \cap A\big) = \sum_{k=1}^{m} \mu\big(E^+_{k,j_0}\big) = \frac{1}{2}\sum_{k=1}^{m} \mu\big(E_{k,j_0}\big) = \frac{1}{2}\sum_{k=1}^{m} \mu\big(C_k \cap D_{j_0}\big) = \frac{1}{2}\mu(D_{j_0}).$$

Analogously, we can prove that $\mu(D_j \cap B) = \frac{1}{2}\mu(D_j)$ for every $j \in \{1,\dots,\ell\}$, which finishes the proof of (ii).

To prove (iii) observe that

$$x_A - x_B = \sum_{k=1}^{m}\sum_{j=1}^{\ell} a_k \left(\mathbf{1}_{E^+_{k,j}} - \mathbf{1}_{E^-_{k,j}}\right) = \sum_{k=1}^{m}\sum_{j=1}^{\ell} a_k u_{k,j}$$

and hence

$$\Big\|T(x_A - x_B)\Big\| \le \sum_{k=1}^{m}\sum_{j=1}^{\ell} |a_k| \|T u_{k,j}\| < 2\varepsilon \,.$$

Therefore, we have that

$$\Big\|T x_A - \frac{1}{2} y\Big\| = \frac{1}{2}\Big\|2T x_A - T x_A - T x_B\Big\| = \frac{1}{2}\Big\|T(x_A - x_B)\Big\| < \varepsilon \,.$$

Analogously, we obtain that $\|T x_B - \frac{1}{2} y\| < \varepsilon$ finishing the proof of (iii). □

Lemma 5.16. *Let* (Ω, Σ, μ) *be an atomless finite measure space,* $1 \le p < \infty$, $T \in \mathcal{L}(L_p(\mu))$ *be a narrow operator, and* $x, y \in L_p(\mu)$ *be simple functions such that* $Tx = y$. *Then for each* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *there exists a partition* $\Omega = A_1 \sqcup \ldots \sqcup A_{2^n}$ *such that for each* $k = 1, \ldots, 2^n$ *we have*

(a) $\|x_{A_k}\|^p = 2^{-n}\|x\|^p$;

(b) $\|y_{A_k}\|^p = 2^{-n}\|y\|^p$;

(c) $\|Tx_{A_k} - 2^{-n}y\| < \varepsilon$.

Proof. Let $y = \sum_{j=1}^{\ell} b_j \mathbf{1}_{D_j}$ for some $b_j \in \mathbb{K}$ and $\Omega = D_1 \sqcup \ldots \sqcup D_\ell$. We proceed by induction on n. Suppose first that $n = 1$ and use Lemma 5.15 to find a partition $\Omega = A \sqcup B$ satisfying properties (i)–(iii). Then (i) and (iii) mean (1) and (3) for $A_1 = A$, $A_2 = B$. Moreover, observe that (2) follows from (ii) :

$$\|y_{A_1}\|^p = \sum_{j=1}^{\ell} |b_j|^p \mu(A_1 \cap D_j) = \sum_{j=1}^{\ell} |b_j|^p \frac{1}{2}\mu(D_j) = \frac{1}{2}\|y\|^p$$

and analogously $\|y_{A_2}\|^p = 2^{-1}\|y\|^p$.

For the induction step suppose that the statement of the lemma is true for $n \in \mathbb{N}$ and find a partition $\Omega = A_1 \sqcup \ldots \sqcup A_{2^n}$ such that for every $k = 1, \ldots, 2^n$ the following hold:

$$\|x_{A_k}\|^p = 2^{-n}\|x\|^p, \quad \|y_{A_k}\|^p = 2^{-n}\|y\|^p, \quad \text{and} \quad \|Tx_{A_k} - 2^{-n}y\| < \varepsilon\,. \tag{5.4}$$

Then, for each $k = 1, \ldots, 2^n$ use Lemma 5.15 for x_{A_k} instead of x, Tx_{A_k} instead of y, the decomposition

$$\Omega = \bigsqcup_{k=1}^{2^n} \bigsqcup_{j=1}^{\ell} (D_j \cap A_k)$$

instead of $\Omega = D_1 \sqcup \ldots \sqcup D_\ell$ and $\frac{\varepsilon}{2}$ instead of ε, and find a partition $\Omega = A(k) \sqcup B(k)$ satisfying properties (i)–(iii) of Lemma 5.15. That is, for each $1 \le k \le 2^n$ we have that:

(i) $\|x_{(A_k \cap A(k))}\|^p = \|x_{(A_k \cap B(k))}\|^p = 2^{-1}\|x_{A_k}\|^p$;

(ii) $\mu(D_j \cap A_k \cap A(k)) = \mu(D_j \cap A_k \cap B(k)) = \frac{1}{2}\mu(D_j \cap A_k)$ for each $j = 1, \ldots, \ell$;

(iii) $\|Tx_{(A_k \cap A(k))} - 2^{-1}Tx_{A_k}\| < \frac{\varepsilon}{2}$ and $\|Tx_{(A_k \cap B(k))} - 2^{-1}Tx_{A_k}\| < \frac{\varepsilon}{2}$.

Let us show that the partition

$$\Omega = \big(A_1 \cap A(1)\big) \sqcup \ldots \sqcup \big(A_{2^n} \cap A(2^n)\big) \sqcup \big(A_1 \cap B(1)\big) \sqcup \ldots \sqcup \big(A_{2^n} \cap B(2^n)\big)$$

has the desired properties for $n + 1$:

Property (1): using (i) and (5.4), one obtains

$$\left\|x_{(A_k \cap A(k))}\right\|^p = \left\|x_{(A_k \cap B(k))}\right\|^p = 2^{-1}\left\|x_{A_k}\right\|^p = 2^{-(n+1)}\|x\|^p .$$

Property (2): for each $k = 1, \ldots, 2^n$ use (ii) and (5.4) to obtain

$$\begin{aligned}\left\|y_{(A_k \cap A(k))}\right\|^p &= \sum_{j=1}^{\ell} |b_j|^p \mu\big(D_j \cap A_k \cap A(k)\big) = \frac{1}{2}\sum_{j=1}^{\ell} |b_j|^p \mu(D_j \cap A_k) \\ &= \frac{1}{2}\left\|y_{A_k}\right\|^p = 2^{-(n+1)}\|y\|^p\end{aligned}$$

and analogously $\|y_{(A_k \cap B)}\|^p = 2^{-(n+1)}\|y\|^p$.
Property (3): for each $k = 1, \ldots, 2^n$ use (iii) and (5.4) to write

$$\begin{aligned}\left\|Tx_{(A_k \cap A(k))} - 2^{-(n+1)}y\right\| &\le \left\|Tx_{(A_k \cap A(k))} - 2^{-1}Tx_{A_k}\right\| + \frac{1}{2}\left\|Tx_{A_k} - 2^{-n}y\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and analogously $\|Tx_{(A_k \cap B(k))} - 2^{-(n+1)}y\| < \varepsilon$, which completes the proof. □

Lemma 5.17. *Let (Ω, Σ, μ) be an atomless finite measure space, $1 \le p < \infty$, let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator, and let $x, y \in L_p(\mu)$ be simple functions such that $Tx = y$. Then for each $n \in \mathbb{N}$, each number λ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \ldots, 2^n - 1\}$ and each $\varepsilon > 0$ there exists a partition $\Omega = A \sqcup B$ such that:*

(a) $\|x_A\|^p = \lambda\|x\|^p$;

(b) $\|y_B\|^p = (1 - \lambda)\|y\|^p$;

(c) $\|Tx_A - \lambda y\| < \varepsilon$.

Proof. Use Lemma 5.16 to choose a partition $\Omega = A_1 \sqcup \ldots \sqcup A_{2^n}$ satisfying properties $(1) - (3)$ with ε/j instead of ε. Let $A = \bigsqcup_{k=1}^{j} A_k$ and $B = \bigsqcup_{k=j+1}^{2^n} A_k$. Then

$$\left\|x_A\right\|^p = \sum_{k=1}^{j}\left\|x_{A_k}\right\|^p = \sum_{k=1}^{j} 2^{-n}\|x\|^p = \lambda\|x\|^p,$$

$$\left\|y_B\right\|^p = \sum_{k=j+1}^{2^n}\left\|y_{A_k}\right\|^p = \sum_{k=j+1}^{2^n} 2^{-n}\|y\|^p = (1-\lambda)\|y\|^p,$$

$$\left\|Tx_A - \lambda y\right\| = \left\|\sum_{k=1}^{j} Tx_{A_k} - \sum_{k=1}^{j} 2^{-n}y\right\| \le \sum_{k=1}^{j}\left\|Tx_{A_k} - 2^{-n}y\right\| < j\frac{\varepsilon}{j} = \varepsilon ,$$

as desired. □

Proof of Theorem 5.14. Let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator of norm one. Fix $\varepsilon > 0, \tau > 0, n \in \mathbb{N}$ and $\lambda \in (0,1)$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \dots, 2^n - 1\}$. Choose a simple function $x \in S_{L_p(\mu)}$ so that $y = Tx$ satisfies $\|y\|^p \geq 1 - \varepsilon$. Without loss of generality we may assume that y is a simple function since T can be approximated by a sequence of narrow operators with the desired property (indeed, take a sequence of simple functions (y_m) converging to y and define $T_m = T - x^\# \otimes (y - y_m)$). Thus $T_m(x) = y_m$, $\|T_m - T\| \leq \|y - y_m\|$, and T_m is narrow for every $m \in \mathbb{N}$, by Proposition 5.5.

Use Lemma 5.17 to find a partition $\Omega = A \sqcup B$ satisfying (a)–(c) and use (b) and (c) to obtain the following estimate:

$$\begin{aligned} \left| \int_B y^\# T x_A \, d\mu - \lambda(1-\lambda)\|y\|^p \right| &= \left| \int_B y^\# T x_A \, d\mu - \lambda \int_B y^\# y \, d\mu \right| \\ &\leq \left\| T x_A - \lambda y \right\| < \varepsilon \, . \end{aligned} \tag{5.5}$$

For $\theta \in \mathbb{T}$, define $z_\theta = \lambda^{-\frac{1}{p}} x_A + \theta(1-\lambda)^{-\frac{1}{p}} \tau y_B$ and observe, using (a) and (b) of Lemma 5.17, that

$$\|z_\theta\|^p = \lambda^{-1} \left\| x_A \right\|^p + (1-\lambda)^{-1} \tau^p \left\| y_B \right\|^p \leq 1 + \tau^p \, .$$

Moreover, using the fact that $(u+v)^\# = u^\# + v^\#$ for disjointly supported elements $u, v \in L_p(\mu)$, it is clear that $z_\theta^\# = \lambda^{-\frac{1}{q}} x_A^\# + \overline{\theta}(1-\lambda)^{-\frac{1}{q}} \tau^{p-1} y_B^\#$. Using this and (5.5) we can write

$$\begin{aligned} (1+\tau^p)v(T) &\geq \max_{\theta \in \mathbb{T}} \left| \int_\Omega z_\theta^\# T z_\theta \, d\mu \right| \qquad (5.6) \\ &= \max_{\theta \in \mathbb{T}} \Bigg| \lambda^{-1} \int_A x^\# T x_A \, d\mu + \theta \lambda^{-\frac{1}{q}} (1-\lambda)^{-\frac{1}{p}} \tau \int_A x^\# T y_B \, d\mu \\ &\qquad + \overline{\theta} \lambda^{-\frac{1}{p}} (1-\lambda)^{-\frac{1}{q}} \tau^{p-1} \int_B y^\# T x_A \, d\mu + (1-\lambda)^{-1} \tau^p \int_B y^\# T y_B \, d\mu \Bigg| \\ &\geq \max_{\theta \in \mathbb{T}} \Bigg| \lambda^{-1} \int_A x^\# T x_A \, d\mu + (1-\lambda)^{-1} \tau^p \int_B y^\# T y_B \, d\mu \\ &\qquad + \theta \lambda^{-\frac{1}{q}} (1-\lambda)^{-\frac{1}{p}} \tau \int_A x^\# T y_B \, d\mu + \overline{\theta} \lambda^{\frac{1}{q}} (1-\lambda)^{\frac{1}{p}} \tau^{p-1} \|y\|^p \Bigg| \\ &\quad - \lambda^{-\frac{1}{p}} (1-\lambda)^{-\frac{1}{q}} \tau^{p-1} \left| \int_B y^\# T x_A \, d\mu - \lambda(1-\lambda)\|y\|^p \right| \\ &\geq \max_{\theta \in \mathbb{T}} \Bigg| \lambda^{-1} \int_A x^\# T x_A \, d\mu + (1-\lambda)^{-1} \tau^p \int_B y^\# T y_B \, d\mu \\ &\qquad + \theta \lambda^{-\frac{1}{q}} (1-\lambda)^{-\frac{1}{p}} \tau \int_A x^\# T y_B \, d\mu + \overline{\theta} \lambda^{\frac{1}{q}} (1-\lambda)^{\frac{1}{p}} \tau^{p-1} \|y\|^p \Bigg| \\ &\quad - \lambda^{-\frac{1}{p}} (1-\lambda)^{-\frac{1}{q}} \tau^{p-1} \varepsilon \end{aligned}$$

$$\begin{aligned}&\geq \max_{\theta\in\mathbb{T}} \left|\theta\lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B \, d\mu + \overline{\theta}\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p\right| \\ &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon.\end{aligned}$$

Let us prove the last step in the formula above. Indeed, we write

$$\begin{aligned}a &= \lambda^{-1}\int_A x^\# T x_A \, d\mu + (1-\lambda)^{-1}\tau^p \int_B y^\# T y_B \, d\mu \\ b &= \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B \, d\mu \\ c &= \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p\end{aligned}$$

and observe that what we need to prove is

$$\max_{\theta\in\mathbb{T}}\left|a+\theta b+\overline{\theta}c\right| \geq \max_{\theta\in\mathbb{T}}\left|\theta b+\overline{\theta}c\right| .$$

This inequality is easy to prove. For a fixed $\theta_0 \in \mathbb{T}$ it is clear that

$$\max_{\theta\in\mathbb{T}}\left|a+\theta b+\overline{\theta}c\right| \geq \max\{\left|a+\left(\theta_0 b+\overline{\theta_0}c\right)\right|, \left|a-\left(\theta_0 b+\overline{\theta_0}c\right)\right|\} \geq \left|\theta_0 b+\overline{\theta_0}c\right|$$

and the arbitrariness of θ_0 gives the desired inequality.

From this point we study the real and the complex case separately. For the complex case, we continue the estimation in (5.6) as follows:

$$\begin{aligned}(1+\tau^p)v(T) &\geq \max_{\theta\in\mathbb{T}}\left|\theta\lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\int_A x^\# T y_B \, d\mu + \overline{\theta}\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p\right| \\ &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\ &= \left|\lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\int_A x^\# T y_B \, d\mu\right| + \left|\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p\right| \\ &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\ &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\ &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}(1-\varepsilon) - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon\,.\end{aligned}$$

By the arbitrariness of ε we can write

$$v(T) \geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\frac{\tau^{p-1}}{1+\tau^p}$$

for every $\tau > 0$ and every $\lambda \in (0,1)$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1,\ldots,2^n-1\}$. Since the dyadic numbers are dense in $[0,1]$ and $\max\limits_{\lambda\in[0,1]} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \kappa_p = \max\limits_{\tau>0}\frac{\tau^{p-1}}{1+\tau^p}$, the last inequality implies $v(T) \geq \kappa_p^2$ which finishes the proof in the complex case.

In the real case, using (a) and (b) of Lemma 5.17, it is easy to check that

$$\begin{aligned}\lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\left|\int_A x^\# T y_B \, d\mu\right| &\leq \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \left\|x_A^\#\right\|_q \left\|y_B\right\|_p \\ &\leq \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \tau\end{aligned}$$

which, together with (5.6) and the choice of y, implies that

$$\begin{aligned}(1+\tau^p)v(T) &\geq \left|\lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\int_A x^\# T y_B \, d\mu + \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p\right| \\ &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\ &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p - \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau\left|\int_A x^\# T y_B \, d\mu\right| \\ &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\ &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}(1-\varepsilon) - \tau - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon\,.\end{aligned}$$

Hence, by the arbitrariness of ε we deduce that

$$v(T) \geq \frac{\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1} - \tau}{1+\tau^p}\,,$$

for every $\tau > 0$ and every $\lambda \in (0,1)$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \ldots, 2^n - 1\}$. Taking supremum over λ, we get

$$v(T) \geq \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p}$$

for each $\tau > 0$, completing the proof. □

Chapter 6

Daugavet-type properties of Lebesgue and Lorentz spaces

The classical theorem of Daugavet [27] asserts that

$$\|I + T\| = 1 + \|T\| \tag{6.1}$$

for every compact operator T on $C[0,1]$, where I is the identity operator. Lozanovskii [85] proved that (6.1) also holds for every compact operator T on $L_1[0,1]$. Equation (6.1) is known as the *Daugavet equation*, and we say that a Banach space X satisfies the *Daugavet property* for a class of operators $\mathcal{M} \subseteq \mathcal{L}(X)$, if (6.1) holds for every operator $T \in \mathcal{M}$. Recently, the study of the Daugavet property for various classes of spaces and operators has attracted a lot of attention (see, e.g. the survey [141]). Daugavet property has important implications for geometry of the space. In particular, a space satisfying the Daugavet property cannot be reflexive. One line of research concerning (6.1) is to find the largest possible class of operators for which (6.1) holds. In particular, Plichko and Popov [110] showed that (6.1) holds for all narrow operators on any atomless space $L_1(\Omega)$. In Section 6.1 we present a further generalization of this result valid for narrow operators on L_1 and replacing the identity operator I with a more general small isomorphism J, see Theorem 6.3.

Another direction of generalizing (6.1) is to consider a weaker inequality. We say that a Banach space X satisfies the *pseudo-Daugavet property* if there exists a strictly increasing function $\delta_X : (0,\infty) \to (0,\infty)$ so that

$$\|I + T\| \geq 1 + \delta_X(\|T\|) \tag{6.2}$$

for every compact operator T on X. Similarly as the Daugavet property, the pseudo-Daugavet property has important geometric implications, in particular it is related to the problem of best compact approximation in X, see [11]. Benyamini and Lin [14] showed that for every p, $1 \leq p < \infty$, $p \neq 2$, $L_p[0,1]$ satisfies (6.2) for compact operators. Plichko and Popov [110] generalized this result to narrow operators, and Boyko and V. Kadets [21] proved that for every $\varepsilon > 0$,

$$\lim_{p \to 1^+} \delta_{L_p}(\varepsilon) = \varepsilon,$$

and thus the Daugavet equation (6.1) on L_1 is a limit case, as p tends to 1, of (6.2) for narrow operators on L_p. We present these results in Section 6.2, Theorems 6.8 and 6.15.

As an immediate corollary of the pseudo-Daugavet property for narrow operators on L_p we obtain that for every projection P onto a rich subspace of L_p, $1 \le p < \infty$, $p \neq 2$,

$$\|P\| \ge 1 + \delta_{L_p}(1)$$

(see Corollary 6.12). This result will be applied in Section 7.5. It answers for L_p a question of Semenov, who asked whether for every separable r.i. function space X, $X \neq L_2$, on $[0,1]$ there exists a constant $k_X > 1$, such that for every rich subspace $Y \subsetneq X$ and every projection $P : X \overset{\text{onto}}{\longrightarrow} Y$, $\|P\| \ge k_X$ (see Open problem 6.14). The authors [118] proved that Semenov's question also has an affirmative answer for projections onto rich subspaces of Lorentz spaces $L_{p,w}[0,1]$, with $p > 2$ and arbitrary weight w. We present this result in Section 6.3, Theorem 6.21.

In Section 6.4 we apply results of previous sections to obtain a theorem in a spirit of the Banach–Stone theorem, but for L_p-spaces. That is, we prove that there exists a constant $k_p > 1$, so that if the Banach–Mazur distance between spaces $L_p(\Omega_1, \Sigma_1, \mu_1)$ and $L_p(\Omega_2, \Sigma_2, \mu_2)$ is less than k_p, then the Maharam sets of the underlying measure spaces coincide.

We note that additional results concerning the Daugavet property of rich subspaces of L_1 and of $C[0,1]$ are presented in Sections 7.6 and 11.3, respectively.

6.1 A generalization of the Daugavet property for L_1 to "small" into isomorphisms instead of the identity

We consider the following question.

Problem 6.1. Given an into isomorphism $J \in \mathcal{L}(L_1)$ and a narrow operator $T \in \mathcal{L}(L_1)$, how can one estimate the value $\|J + T\|$ from below?

A simple example shows that we cannot estimate this norm by the sum of $\|J\|$ and another nonnegative summand.

Example 6.2. For every $\varepsilon > 0$ there exist an into isomorphism $J \in \mathcal{L}(L_1)$ with $\|J\| = 1 + \varepsilon$ and $\|J^{-1}\| = 1$, and a narrow operator $T \in \mathcal{L}(L_1)$ with $\|T\| = \varepsilon$, such that $\|J + T\| = 1$.

Construction. Let $T \in \mathcal{L}(L_1)$ be any narrow operator satisfying $\|T\| = \varepsilon$ and $\operatorname{supp} Tx \subseteq [1/2, 1]$ for each $x \in L_1$. We set

$$Jx(t) = -Tx(t) + \begin{cases} 2x(2t), \text{if } t \in [0, 1/2), \\ 0, \text{if } t \in [1/2, 1], \end{cases}$$

for each $x \in L_1$. □

In Example 6.2, $\|J + T\| < \|J\|$. But if we estimate the norm $\|J + T\|$ as

$$\|J + T\| \geq \|T\| + \varphi(d)$$

where $d = \|J\|\|J^{-1}\|$, then a positive value for $\varphi(d)$ for $d < 2$ is obtained. Our main result is the following.

Theorem 6.3. *Let $T \in \mathcal{L}(L_1)$ be a narrow operator and let $J \in \mathcal{L}(L_1)$ be an into isomorphism with $d = \|J\|\|J^{-1}\| < 2$. Then*

$$\|J + T\| \geq \|T\| + \|J\| \left(\frac{2}{d} - 1 \right).$$

As Example 6.2 shows, the estimate obtained in Theorem 6.3 is exact. When $J = I$ we obtain the following corollary.

Corollary 6.4. *Let (Ω, Σ, μ) be a finite atomless measure space. Then $L_1(\mu)$ satisfies the Daugavet property for narrow operators.*

We note that exact set all of operators on L_1 that satisfy the Daugavet equation is larger than the set of narrow operators, and was described by Shvidkoy [131].

For the proof of Theorem 6.3 we will need several lemmas.

Auxiliary lemmas

We formulate the Enflo–Rosenthal truncation lemma for the space L_1 only. Following [4], a sequence $(x_n)_1^\infty$ in a Banach space X is called *colacunary* if there is a number $\theta > 0$ such that

$$\Big\| \sum_{k=1}^{n} a_k x_k \Big\| \geq \theta \sum_{k=1}^{n} |a_k|$$

for each n and each collection of scalars $(a_k)_1^n$. In this case we say that $(x_n)_1^\infty$ is colacunary with the constant θ. Note that a normalized sequence is colacunary with a constant $\theta \in (0, 1)$ if and only if it is θ^{-1}-equivalent to the unit vector basis of ℓ_1.

The following lemma of Enflo and Rosenthal [36] was stated for normalized sequences (x_n). However, the same proof shows that it is true for arbitrary bounded sequences.

Lemma 6.5. *Let $(x_n)_1^\infty$ be a bounded colacunary sequence in L_1 with a constant $\theta \in (0, 1]$. Then for each $\delta \in (0, \theta)$ and each number $M > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have*

$$\| {}^M x_n - x_n \| > \delta, \quad \textit{where } \; x(t) = \begin{cases} x(t) & \textit{if } |x(t)| \leq M, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose this were false. Then, since (x_n) is bounded in L_1, by passing to a subsequence of the x_ns if necessary, we may assume without loss of generality that for any $\delta \in (0,\theta)$, there is $M > 0$ such that $\| {}^M x_n - x_n \| \le \delta$ for each $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and observe that by the triangle inequality,

$$\int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s)\big({}^M x_k - x_k\big) \Big\| \, ds \le \sum_{k=1}^{n} \| {}^M x_k - x_k \| \le \delta n \, , \tag{6.3}$$

where (r_k) is the Rademacher system. By colacunarity, $\| \sum_{k=1}^n r_k(s) x_k \| \ge \theta n$ for each $s \in [0,1]$. Hence,

$$\int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s) x_k \Big\| \, ds \ge \theta n \, . \tag{6.4}$$

By the triangle inequality in $L_1[0,1]^2$,

$$\begin{aligned} \int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s) x_k \Big\| \, ds - \int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s)\big({}^M x_k - x_k\big) \Big\| \, ds \\ \le \int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s) {}^M x_k \Big\| \, ds \, . \end{aligned} \tag{6.5}$$

Combining (6.3), (6.4) and (6.5), we get

$$n\theta\Big(1 - \frac{\delta}{\theta}\Big) = n\theta - n\delta \le \int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s)\, {}^M x_k \Big\| \, ds \, . \tag{6.6}$$

On the other hand,

$$\begin{aligned} \int_{[0,1]} \Big\| \sum_{k=1}^{n} r_k(s)\, {}^M x_k \Big\| \, ds &\overset{\text{by Fubini's theorem}}{=} \iint_{[0,1]^2} \Big| \sum_{k=1}^{n} r_k(s)\, {}^M x_k(t) \Big| \, ds\, dt \\ &\overset{\text{by Hölder's inequality}}{\le} \left(\iint_{[0,1]^2} \Big| \sum_{k=1}^{n} r_k(s)\, {}^M x_k(t) \Big|^2 \, ds\, dt \right)^{1/2} \\ &\overset{\text{by orthogonality of } r_k' s}{=} \left(\int_{[0,1]} \sum_{k=1}^{n} \Big| {}^M x_k(t) \Big|^2 \, ds\, dt \right)^{1/2} \le M\sqrt{n}. \end{aligned}$$

Combining (6.6) with the last inequality, we obtain

$$\sqrt{n}\theta\Big(1 - \frac{\delta}{\theta}\Big) \le M \, ,$$

that cannot be true for all $n \in \mathbb{N}$. □

Lemma 6.6. *Let $T \in \mathcal{L}(L_1)$. Then for every $\varepsilon > 0$ there exists $A \in \Sigma^+$ such that $\|T\mathbf{1}_A\| \geq (\|T\| - \varepsilon)\mu(A)$.*

Proof. Given $\varepsilon > 0$, we choose a simple function $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$ where $A_1 \sqcup \ldots \sqcup A_m = [0,1]$ and $A_k \in \Sigma^+$ for $k = 1, \ldots, m$, so that $\|Tx\| \geq (\|T\| - \varepsilon)\|x\|$. We claim that $\|T\mathbf{1}_{A_k}\| \geq (1 - \varepsilon)\mu(A_k)$ for some $k \in \{1, \ldots, m\}$. Indeed, otherwise

$$\|Tx\| \leq \sum_{k=1}^m |a_k| \|T\mathbf{1}_{A_k}\| < \sum_{k=1}^m |a_k| (\|T\| - \varepsilon)\mu(A) = \big(\|T\| - \varepsilon\big)\|x\|,$$

which is a contradiction. □

Note that Lemma 6.6 is a weak version of Theorem 7.31.

Lemma 6.7. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in Σ^+ with $\lim_{n\to\infty} \mu(A_n) = 0$, and $x_n = \mathbf{1}_{A_n}/\mu(A_n)$ for each $n \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists a subsequence $(x_{i(n)})_{n=1}^\infty$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1.*

Proof. Since (x_n) is a normalized sequence in L_1, our goal is to construct a colacunary subsequence with constant $(1+\varepsilon)^{-1}$. By arbitrariness of $\varepsilon > 0$, this is equivalent to constructing a colacunary subsequence with constant $1 - \varepsilon$, for any $\varepsilon > 0$. Fix any $\varepsilon \in (0,1)$ and $\alpha \in (0,1)$ so that $\frac{\alpha}{1-\alpha} \leq \frac{\varepsilon}{2}$. Since $\lim_{n\to\infty} \mu(A_n) = 0$, there exists a subsequence of (A_n) which, for convenience of the notation, we also denote by (A_n), such that $\mu(A_{n+1}) \leq \alpha\mu(A_n)$ for each $n \in \mathbb{N}$.

We show that (x_n) is a colacunary sequence with constant $1 - \varepsilon$. Indeed, we have

$$\begin{aligned} \mu\Big(\bigcup_{k=n+1}^\infty A_k\Big) &\leq \sum_{k=n+1}^\infty \mu(A_k) \leq \mu(A_n)\Big(\alpha + \alpha^2 + \alpha^3 + \ldots\Big) \\ &= \mu(A_n)\,\frac{\alpha}{1-\alpha} \leq \frac{\varepsilon}{2}\,\mu(A_n). \end{aligned} \tag{6.7}$$

Set $B_n = A_n \setminus \bigcup_{k=n+1}^\infty A_k$ and $y_n = \frac{\mathbf{1}_{B_n}}{\mu(B_n)}$ for each $n \in \mathbb{N}$, and observe that for each $n \in \mathbb{N}$

$$\begin{aligned} \|x_n - y_n\| &= \left(\frac{1}{\mu(B_n)} - \frac{1}{\mu(A_n)}\right)\mu(B_n) + \frac{1}{\mu(A_n)}\mu\Big(\bigcup_{k=n+1}^\infty A_k\Big) \\ &= \frac{1}{\mu(A_n)}\left(\mu(A_n) - \mu(B_n) + \mu\Big(\bigcup_{k=n+1}^\infty A_k\Big)\right) = \frac{2\mu\Big(\bigcup_{k=n+1}^\infty A_k\Big)}{\mu(A_n)} \\ &\overset{\text{by (6.7)}}{\leq} \varepsilon. \end{aligned}$$

Since (B_n) is a sequence of disjoint sets, the sequence y_n is isometrically equivalent to the unit vector basis of ℓ_1. Thus for any finite collection of scalars $(a_k)_{k=1}^n$ we have

$$\Big\|\sum_{k=1}^n a_k x_k\Big\| \ge \Big\|\sum_{k=1}^n a_k y_k\Big\| - \sum_{k=1}^n |a_k|\|x_k - y_k\| \ge (1-\varepsilon)\sum_{k=1}^n |a_k| . \qquad \square$$

Proof of Theorem 6.3

Let $\varepsilon > 0$. By Lemma 6.6, there exists $A \in \Sigma^+$ so that for $x = \mathbf{1}_A/\mu(A)$ we have $\|x\| = 1$ and $\|Tx\| \ge \|T\| - \varepsilon$. For any $n \in \mathbb{N}$, by Lemma 1.11, there exists a decomposition $A = A_n' \sqcup A_n''$ so that $\mu(A_n'') = 2^{-n}\mu(A)$ and $\|Th_n\| < \varepsilon\mu(A)$ where $h_n = \mathbf{1}_{A_n'} - (2^n - 1)\mathbf{1}_{A_n''}$. Note that for every $n \in \mathbb{N}$ we have

$$x - \frac{h_n}{\mu(A)} = \frac{2^n}{\mu(A)}\mathbf{1}_{A_n''} = \frac{\mathbf{1}_{A_n''}}{\mu(A_n'')} ,$$

and $\|x - h_n/\mu(A)\| = 1$. By Lemma 6.7, the normalized sequence $(x - h_n/\mu(A))_{n=1}^\infty$ contains a subsequence $(x - h_{i(n)}/\mu(A))_{n=1}^\infty$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1.

Suppose that $\|J\| = 1$. Then $\|J^{-1}\| \ge 1$ and $\theta = \|J^{-1}\|^{-1}(1+\varepsilon)^{-1} \le 1$. Putting $x_n = J(x - h_{i(n)}/\mu(A))$, we obtain that for any n and any n-tuple of scalars $(a_k)_{k=1}^n$

$$\begin{aligned}\Big\|\sum_{k=1}^n a_k x_k\Big\| &= \Big\|J\sum_{k=1}^n a_k\Big(x - \frac{h_{i(k)}}{\mu(A)}\Big)\Big\| \\ &\ge \frac{1}{\|J^{-1}\|}\Big\|\sum_{k=1}^n a_k\Big(x - \frac{h_{i(k)}}{\mu(A)}\Big)\Big\| \ge \theta\sum_{k=1}^n |a_k| .\end{aligned}$$

Let $\nu > 0$ be so that for each $B \in \Sigma^+$, if $\mu(B) < \nu$ then $\int_B |Tx|\,\mathrm{d}\mu < \varepsilon$. Let $M = 2\nu^{-1}$. By Lemma 6.5 there exists $n \in \mathbb{N}$ such that $\|^M x_n - x_n\| > \theta - \varepsilon$. Let $B = \{t \in [0,1] : |x_n(t)| > M\}$. Then

$$\mu(B) \le \frac{1}{M}\int_B |x_n|\,\mathrm{d}\mu \le \frac{1}{M}\|J\|\|x - \frac{h_n}{\mu(A)}\| = \frac{1}{M} < \nu .$$

Hence,

$$\begin{aligned}\Big\|(J+T)\Big(x - \frac{h_{i(n)}}{\mu(A)}\Big)\Big\| &= \Big\|x_n + Tx - \frac{Th_{i(n)}}{\mu(A)}\Big\| \ge \|x_n + Tx\| - \varepsilon \\ &= \big\|(x_n + Tx)|_B\big\| + \big\|(x_n + Tx)|_{[0,1]\setminus B}\big\| - \varepsilon \\ &\ge \big\|x_n|_B\big\| - \big\|Tx|_B\big\| + \big\|Tx|_{[0,1]\setminus B}\big\| - \big\|x_n|_{[0,1]\setminus B}\big\| - \varepsilon \\ &\ge \theta - \varepsilon - \varepsilon + \big\|Tx - Tx|_B\big\| - \|x_n\| + \big\|x_n|_B\big\| - \varepsilon \\ &\ge \theta + \|Tx\| - \varepsilon - 1 + \theta - \varepsilon - 3\varepsilon \ge 2\theta + \|T\| - 1 - 6\varepsilon\end{aligned}$$

and by arbitrariness of $\varepsilon > 0$ we obtain

$$\|J + T\| \geq \frac{2}{\|J^{-1}\|} + \|T\| - 1 .$$

Suppose now there is no restriction on $\|J\|$. Since

$$\left\| \Big(\frac{J}{\|J\|} \Big)^{-1} \right\| = \Big\| \|J\| J^{-1} \Big\| = \|J\| \|J^{-1}\| ,$$

we have

$$\begin{aligned} \|J + T\| &= \|J\| \left\| \frac{J}{\|J\|} + \frac{T}{\|J\|} \right\| \geq \|J\| \Big(\frac{2}{\|J\| \|J^{-1}\|} - 1 + \left\| \frac{T}{\|J\|} \right\| \Big) \\ &= \|T\| + \|J\| \Big(\frac{2}{d} - 1 \Big) . \end{aligned}$$

6.2 Pseudo-Daugavet property for narrow operators on L_p, $p \neq 2$

Benyamini and Lin [14] proved that for each $p \in (1,2) \cup (2, +\infty)$ and each $\varepsilon > 0$ there exists $\delta_p(\varepsilon) > 0$ such that for every compact operator $K \in \mathcal{L}(L_p)$ of norm $\|K\| \geq \varepsilon$ we have

$$\|I + K\| \geq 1 + \delta_p(\varepsilon).$$

This result was generalized to narrow operators in [110, p. 64] (see Theorem 6.8). As in the case of compact operators, the proof for narrow operators in the case $1 < p < 2$ is much more involved than that for $2 < p < \infty$. The essential difference for narrow operators is that we cannot consider the case $2 < p < \infty$ only, and then use the duality argument, since, by Corollary 5.10, the conjugate to a narrow operator need not be narrow.

Boyko and V. Kadets [21] proved that for every $\varepsilon > 0$, $\delta_p(\varepsilon)$ tends to ε, as p goes to 1 (see Theorem 6.15).

This section is devoted to the proofs of these results.

Pseudo-Daugavet property for narrow operators on L_p, $p \neq 2$

The main result of this subsection is the following theorem from [110].

Theorem 6.8. *Let $1 < p < \infty$, $p \neq 2$ and (Ω, Σ, μ) be a finite atomless measure space. Then there exists an increasing function $\delta_p : (0, \infty) \to (0, \infty)$ so that for every narrow operator $T \in \mathcal{L}(L_p(\mu))$ we have*

$$\|I + T\| \geq 1 + \delta_p(\|T\|) .$$

For the proof of Theorem 6.8 we need some lemmas.

Lemma 6.9. *Let* $1 < p < \infty, \quad p \neq 2$. *Define*

$$h_0 = 3 \cdot \mathbf{1}_{\left[0,\frac{1}{4}\right]} - \mathbf{1}_{\left(\frac{1}{4},1\right]} \quad \textit{and} \quad \mathbf{1} = \mathbf{1}_{[0,1]} \,.$$

Then there exists a number $\alpha_p \neq 0$, *and an increasing function* $\beta_p(\Delta) > 0$ *defined for each* $\Delta > 0$, *such that for any* $t \in \mathbb{R}$ *the inequality* $|t-1| \geq \Delta$ *implies*

$$\left\|\mathbf{1} - \alpha_p h_0\right\|^p + \beta_p(\Delta) \cdot |t-1|^p \leq \left\|t\mathbf{1} - \alpha_p h_0\right\|^p .$$

Using the idea of [14], one can show that for $2 < p < \infty$ a number $\beta_p(\Delta)$ can be chosen independently of Δ, but for the case $1 < p < 2$ this is impossible.

Proof of Lemma 6.9. Denote by γ_p the scalar that minimizes the function

$$f(\gamma) = \|\gamma\mathbf{1} - h_0\|^p = \frac{1}{4}\left|3-\gamma\right|^p + \frac{3}{4}\left|\gamma+1\right|^p .$$

We show that $\gamma_p \neq 0$. For this purpose, it is enough to consider the function $f(\gamma)$ on the segment $[-1, 1]$, where it has the following form:

$$f(\gamma) = \frac{1}{4}\left(3-\gamma\right)^p + \frac{3}{4}\left(\gamma+1\right)^p .$$

Then

$$\frac{4}{p}\, f'(\gamma) = -(3-\gamma)^{p-1} + 3\,(\gamma+1)^{p-1} ,$$

and hence, $\frac{4}{p}\, f'(0) = -3^{p-1} + 3 \neq 0$ for $p \neq 2$. Thus, 0 cannot be a point of minimum for $f(\gamma)$.

Let $\alpha_p = \gamma_p^{-1}$. Then for each $t \in \mathbb{R}$,

$$\|\mathbf{1} - \alpha_p h_0\| \leq \|t\mathbf{1} - \alpha_p h_0\| . \tag{6.8}$$

Put

$$\beta_p(\Delta) = \inf_{|t-1|\geq\Delta} \left\{|t-1|^{-p}\left(\|t\mathbf{1} - \alpha_p h_0\|^p - \|\mathbf{1} - \alpha_p h_0\|^p\right)\right\} . \tag{6.9}$$

Clearly $\beta_p(\Delta)$ is an increasing function of Δ. It remains to prove that $\beta_p(\Delta) \neq 0$ for all $\Delta > 0$. Denote by $\psi(t)$ the expression in the braces in (6.9). First we prove that

$$\lim_{|t-1|\to\infty} \psi(t) = 1 . \tag{6.10}$$

Indeed, by the triangle inequality,

$$\|(t-1)\,\mathbf{1}\| - \|\mathbf{1} - \alpha_p h_0\| \leq \|t\mathbf{1} - \alpha_p h_0\| \leq \|(t-1)\,\mathbf{1}\| + \|\mathbf{1} - \alpha_p h_0\| .$$

Hence,

$$\lim_{|t-1|\to\infty} \frac{\|t\mathbf{1} - \alpha_p h_0\|}{|t-1|} = \lim_{|t-1|\to\infty} \frac{\|(t-1)\mathbf{1}\|}{|t-1|} = 1 .$$

Thus,

$$\lim_{|t-1|\to\infty} \psi(t) = \lim_{|t-1|\to\infty} \frac{\|t\mathbf{1} - \alpha_p h_0\|^p}{|t-1|^p} - \lim_{|t-1|\to\infty} \frac{\|\mathbf{1} - \alpha_p h_0\|^p}{|t-1|^p} = 1\,.$$

By the continuity of the nonnegative function $\psi(t)$ (see (6.8)), and by (6.10) we deduce that, if $\beta_p(\Delta) = 0$, then there would exist $t_0 \neq 1$ so that $\psi(t_0) = 0$, that is,

$$\|\mathbf{1} - \alpha_p h_0\| = \|t_0 \mathbf{1} - \alpha_p h_0\|\,.$$

The strict convexity of $L_p(\mu)$ for $p > 1$ [25] means that if $x, y \in L_p(\mu)$, $x \neq y$ and $\|x\| = \|y\| > 0$, then $\|(x+y)/2\| < \|y\|$. In particular, for $x = t_0\mathbf{1} - \alpha_p h_0$ and $y = \mathbf{1} - \alpha_p h_0$ we would obtain

$$\left\| \frac{t_0 + 1}{2} \mathbf{1} - \alpha_p h_0 \right\| < \|\mathbf{1} - \alpha_p h_0\|\,,$$

which contradicts (6.8). This implies that the assumption $\beta_p(\Delta) = 0$ is false. □

Lemma 6.10. *For all numbers $a, b \geq 0$ the following inequality holds*

$$a^p \leq b^p + p \cdot |a - b| \cdot \max\{a^{p-1}, b^{p-1}\}\,.$$

Proof of Lemma 6.10. We consider two cases.

(1) $0 \leq a \leq b$. We need to prove that $a^p \leq b^p + p(b-a)b^{p-1}$. Consider the function $\varphi(a) = b^p + p(b-a)b^{p-1} - a^p$ on $[a, b]$. Since $\varphi'(a) = -pb^{p-1} - pa^{p-1} \leq 0$ on $[a, b]$, we have that $\varphi(a) \geq \varphi(b) = 0$.

(2) $0 \leq b \leq a$. Now we need to prove that $a^p \leq b^p + p(a-b)a^{p-1}$. We consider the function $\psi(b) = b^p + p(a-b)a^{p-1} - a^p$ on $[b, a]$. Since $\psi'(b) = pb^{p-1} - pa^{p-1} \leq 0$ on $[b, a]$, we have that $\psi(b) \geq \psi(a) = 0$. □

Lemma 6.11. *Let $1 < p < \infty$, $p \neq 2$. There are numbers $\nu_p > 0$ and $\xi_p > 0$, and an increasing function $\eta_p(\Delta) > 0$ defined for every $\Delta > 0$, such that if $T \in \mathcal{L}(L_p(\mu))$ is a narrow operator and $x, y \in L_p(\mu)$ satisfy $\|x\| = 1$ and $\|y\| \geq \Delta$, then for every $\varepsilon > 0$, there exists $h_\varepsilon \in L_p(\mu)$ such that*

$$\|Th_\varepsilon\| < \varepsilon, \quad \nu_p \leq \|x - h_\varepsilon\| \leq 1 + \xi_p$$

and

$$\|x - h_\varepsilon\|^p + \eta_p(\Delta)\, \|y\|^p \leq \|x + y - h_\varepsilon\|^p\,.$$

Proof of Lemma 6.11. Let

$$\nu_p = \frac{\|\mathbf{1} - \alpha_p h_0\|}{2}, \quad \xi_p = |\alpha_p| \left(\frac{3^p + 3}{4}\right)^{\frac{1}{p}}, \quad \eta_p(\Delta) = \frac{1}{4}\beta_p\left(\frac{\Delta}{2^{1/p}}\right),$$

where $h_0, \alpha_p, \beta_p(\Delta)$ are defined in Lemma 6.9. Since the function β_p is increasing, so is the function η_p.

Let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator, $x, y \in L_p(\mu)$, $\|x\| = 1$, $\|y\| \geq \Delta$, and $\varepsilon > 0$. Choose simple functions $\hat{x}, \hat{y} \in L_p(\mu)$ so that:

(a) The term $\hat{x}(\omega) \neq 0$ almost everywhere; and, $\hat{x}$ and $\hat{y}$ take values $a_k \neq 0$ and b_k, respectively, on disjoint subsets A_k. Moreover, $\Omega = \bigsqcup_{k=1}^{m} A_k$.

(b) $\|x - \hat{x}\| \leq \nu_p$, and

$$\max\left\{\|x-\hat{x}\|(1+\xi_p)^{p-1}, \left(\|x-\hat{x}\|+\|y-\hat{y}\|\right)\left(1+\|y\|+\xi_p\right)^{p-1}\right\} \leq \frac{1}{2p}\eta_p(\Delta)\|y\|^p.$$

(c) $\|\hat{x}\| = \|x\| = 1$, $\|\hat{y}\| = \|y\|$.

Fix any $k \in \mathbb{N}$, $1 \leq k \leq m$ and choose, using Lemma 1.11, a function $h_k \in L_p(\mu)$ so that

$$h_k = 3 \cdot \mathbf{1}_{A_k'} - \mathbf{1}_{A_k''}, \quad A_k' \sqcup A_k'' = A_k, \quad \mu(A_k') = \frac{\mu(A_k)}{4}$$

and

$$\|Th_k\| < \frac{\varepsilon}{2m|a_k||\alpha_p|}.$$

By Lemma 6.9 for $t = 1 + b_k/a_k$, if $|b_k/a_k| \geq \Delta_1$ then

$$\left\|a_k \mathbf{1}_{A_k} - a_k\alpha_p h_k\right\|^p + \beta_p(\Delta_1)\|b_k\mathbf{1}_{A_k}\|^p \leq \left\|(a_k + b_k)\,\mathbf{1}_{A_k} - a_k\alpha_p h_k\right\|^p. \tag{6.11}$$

The definition of α_p implies that for each $k = 1, \dots, m$ the following inequality holds

$$\left\|a_k \mathbf{1}_{A_k} - a_k\alpha_p h_k\right\|^p \leq \left\|(a_k + b_k)\,\mathbf{1}_{A_k} - a_k\alpha_p h_k\right\|^p. \tag{6.12}$$

Set

$$y_1(\omega) = \begin{cases} \hat{y}(\omega) & \text{if } \left|\frac{\hat{y}(\omega)}{\hat{x}(\omega)}\right| \geq \frac{\|\hat{y}\|}{2^{1/p}} \\ 0 & \text{otherwise} \end{cases}$$

and $y_2 = \hat{y} - y_1$. Since y_1 and y_2 have disjoint supports, $\|\hat{y}\|^p = \|y_1\|^p + \|y_2\|^p$. And since

$$|y_2(\omega)| \leq \frac{\|\hat{y}\|}{2^{1/p}} \cdot |\hat{x}(\omega)|$$

almost everywhere, we have $\|y_2\| \leq \|\hat{y}\|/2^{1/p}$, and hence

$$\|y_1\|^p \geq \|\hat{y}\|^p - \frac{\|\hat{y}\|^p}{2} = \frac{\|\hat{y}\|^p}{2}. \tag{6.13}$$

Decompose the set $\{1, \dots, m\}$ into two disjoint subsets M_1 and M_2, so that M_1 consists exactly of those k, for which $|b_k/a_k| \geq \|\hat{y}\|/2^{1/p}$.

We set in (6.11) the value $\Delta_1 = \Delta/2^{1/p}$ and observe that (6.11) holds for all $k \in M_1$. Then we add inequalities (6.11) for all $k \in M_1$, and inequalities (6.12) for all $k \in M_2$:

$$\left\|\hat{x} - \sum_{k=1}^{m} a_k \alpha_p h_k\right\|^p + \beta_p\left(\frac{\Delta}{2^{1/p}}\right) \sum_{k \in M_1} \|b_k \mathbf{1}_{A_k}\|^p \leq \left\|\hat{x} + \hat{y} - \sum_{k=1}^{m} a_k \alpha_p h_k\right\|^p .$$

By (6.13), we obtain

$$\sum_{k \in M_1} \|b_k \mathbf{1}_{A_k}\|^p = \|y_1\|^p \geq \frac{\|\hat{y}\|^p}{2} .$$

Next make

$$h_\varepsilon = \alpha_p \sum_{k=1}^{m} a_k h_k .$$

Then

$$\begin{aligned} \|\hat{x} - h_\varepsilon\|^p + \frac{1}{2}\beta_p\left(\frac{\Delta}{2^{1/p}}\right)\|\hat{y}\|^p &\leq \|\hat{x} - h_\varepsilon\|^p + \beta_p\left(\frac{\Delta}{2^{1/p}}\right) \sum_{k \in M_1} \|b_k \mathbf{1}_{A_k}\|^p \\ &\leq \|\hat{x} + \hat{y} - h_\varepsilon\|^p. \end{aligned} \tag{6.14}$$

We show that h_ε satisfies the assertions of the lemma. Indeed,

$$\|T h_\varepsilon\| \leq |\alpha_p| \sum_{k=1}^{m} |a_k| \|T h_k\| < \varepsilon ,$$

by the choice of h_k, and since $\|\mathbf{1}_{A_k} - \alpha_p h_k\|^p = \mu(A_k)\|\mathbf{1} - \alpha_p h_0\|^p$, using the definitions of h_ε and ν_p together with condition (a), we obtain

$$\begin{aligned} \|x - h_\varepsilon\| &\geq \|\hat{x} - h_\varepsilon\| - \|x - \hat{x}\| \\ &= \Big(\sum_{k=1}^{m} \|a_k \mathbf{1}_{A_k} - \alpha_p a_k h_k\|^p\Big)^{\frac{1}{p}} - \|x - \hat{x}\| \\ &\geq \Big(\sum_{k=1}^{m} \|a_k \mathbf{1}_{A_k} - \alpha_p a_k h_k\|^p\Big)^{\frac{1}{p}} - \nu_p \\ &= \Big(\sum_{k=1}^{m} |a_k|^p \mu(A_k) \|\mathbf{1} - \alpha_p h_0\|^p\Big)^{\frac{1}{p}} - \nu_p \\ &= \|\mathbf{1} - \alpha_p h_0\| \cdot \|\hat{x}\| - \nu_p = 2\nu_p - \nu_p = \nu_p . \end{aligned}$$

Then, by Lemma 6.10, we estimate

$$
\begin{aligned}
&\|x-h_\varepsilon\|^p+\eta_p(\Delta)\|y\|^p-\|x+y-h_\varepsilon\|^p\\
&\le \|\hat{x}-h_\varepsilon\|^p+2\eta_p(\Delta)\|y\|^p-\|\hat{x}+\hat{y}-h_\varepsilon\|^p\\
&-\eta_p(\Delta)\|y\|^p+p\cdot\big|\|x-h_\varepsilon\|-\|\hat{x}-h_\varepsilon\|\big|\cdot\max\big\{\|x-h_\varepsilon\|^{p-1}+\|\hat{x}-h_\varepsilon\|^{p-1}\big\}\\
&+p\cdot\big|\|x+y-h_\varepsilon\|-\|\hat{x}+\hat{y}-h_\varepsilon\|\big|\cdot\max\big\{\|x+y-h_\varepsilon\|^{p-1},\|\hat{x}+\hat{y}-h_\varepsilon\|^{p-1}\big\}
\end{aligned}
$$

(using (6.14) and (c))

$$
\begin{aligned}
&\le -\eta_p(\Delta)\,\|y\|^p+p\,\|x-\hat{x}\|\max\big\{\|x-h_\varepsilon\|^{p-1},\|\hat{x}-h_\varepsilon\|^{p-1}\big\}\\
&\quad+p\,\big(\|x-\hat{x}\|+\|y-\hat{y}\|\big)\max\big\{\|x+y-h_\varepsilon\|^{p-1},\|\hat{x}+\hat{y}-h_\varepsilon\|^{p-1}\big\}\,. \qquad (6.15)
\end{aligned}
$$

By definition of h_ε and ξ_p, we have that

$$
\|h_\varepsilon\|=|\alpha_p|\Big(\sum_{k=1}^m|a_k|^p\Big(3^p\cdot\frac{1}{4}\mu(A_k)+\frac{3}{4}\mu(A_k)\Big)\Big)^{1/p}=\xi_p\,\|\hat{x}\|=\xi_p
$$

and hence, $\|x-h_\varepsilon\|\le\|x\|+\|h_\varepsilon\|=1+\xi_p$. Finally, from (6.15), (b) and (c) we deduce

$$
\begin{aligned}
&\|x-h_\varepsilon\|^p+\eta_p(\Delta)\|y\|^p-\|x+y-h_\varepsilon\|^p\\
&\le -\eta_p(\Delta)\|y\|^p+p\|x-\hat{x}\|(1+\xi_p)^{p-1}+p\big(\|x-\hat{x}\|+\|y-\hat{y}\|\big)(1+\|y\|+\xi_p)^{p-1}\\
&\le 0\,.
\end{aligned}
$$

□

Proof of Theorem 6.8. Let $T\in\mathcal{L}(L_p(\mu))$ be a narrow operator, $\|T\|>0$, $0<\varepsilon<\|T\|/2$, and $x\in L_p(\mu)$ so that $\|x\|=1$ and $\|Tx\|\ge\|T\|-\varepsilon$. Set $y=Tx$ and choose, by Lemma 6.11, $h\in L_p(\mu)$ so that $\|Th\|<\varepsilon$, $\nu_p\le\|x-h\|\le 1+\xi_p$ and

$$
\|x-h\|^p+\eta_p\Big(\frac{\|T\|}{2}\Big)\,\|y\|^p\le\|x+y-h\|^p\,. \qquad (6.16)
$$

Then

$$
\begin{aligned}
\frac{\|(I+T)(x-h)\|}{\|x-h\|}&\ge\frac{\|x+y-h\|}{\|x-h\|}-\frac{\|Th\|}{\|x-h\|}\\
&\ge\frac{\|x+y-h\|}{\|x-h\|}-\frac{\varepsilon}{\nu_p}\,. \qquad (6.17)
\end{aligned}
$$

Using (6.16), we obtain

$$
\begin{aligned}
\frac{\|x+y-h\|^p}{\|x-h\|^p}&\ge 1+\eta_p\Big(\frac{\|T\|}{2}\Big)\frac{\|y\|^p}{\|x-h\|^p}\\
&\ge 1+\eta_p\Big(\frac{\|T\|}{2}\Big)\frac{(\|T\|-\varepsilon)^p}{(1+\xi_p)^p}\,. \qquad (6.18)
\end{aligned}
$$

By arbitrariness of $\varepsilon \in (0, \|T\|/2)$, (6.17) and (6.18) we obtain

$$\|I + T\| \geq \Big(1 + \eta_p\Big(\frac{\|T\|}{2}\Big)\frac{\|T\|^p}{(1+\xi_p)^p}\Big)^{1/p} = 1 + \delta_p(\|T\|)\,. \qquad \square$$

Corollary 6.12 ([114]). *Let $1 \leq p < \infty$ and $p \neq 2$. Then there exists a constant $k_p > 1$, $k_1 = 2$, such that for every finite atomless measure space (Ω, Σ, μ), if $P \neq I$ is a projection from $L_p(\Omega, \Sigma, \mu)$ onto a rich subspace, then $\|P\| \geq k_p$.*

Proof. For $p \in (1, 2) \cup (2, +\infty)$ we set $k_p = 1 + \delta_p(1)$, where $\delta_p(1)$ satisfies Theorem 6.8. If $P \neq I$ is a projection onto a rich subspace of $L_p(\mu)$ then $T = I - P$ is a narrow projection, which is nonzero since $P \neq I$. Hence, by Theorem 6.8 ,

$$\|P\| = \|I - T\| \geq 1 + \delta_p(\|T\|) \geq 1 + \delta_p(1) = k_p\,.$$

For $p = 1$, we use the Daugavet property (see Corollary 6.4) and the same argument with $k_1 = 2$. $\square$

Franchetti [43] proved that the exact value of the constant k_p is equal to $\|I - A\|_p$, where A is the rank-one projection defined by

$$Ax \stackrel{\text{def}}{=} \Big(\int_\Omega x(s)d\mu(s)\Big)\cdot\mathbf{1}\,.$$

Note that the projection A is well defined in any r.i. space with finite measure. The exact value of $\|I - A\|_p$ for $p \in (1, \infty)$, has been evaluated by Franchetti [43] and, independently, by Oskolkov (unpublished).

A classical Ando's theorem [9] says that a subspace X of an $L_p(\mu)$-space is isometrically isomorphic to an $L_p(\nu)$ space if and only if it is 1-complemented in $L_p(\mu)$. Thus, we have the following consequence of Corollary 6.12.

Corollary 6.13. *Let (Ω, Σ, μ) be an atomless measure space and $1 \leq p < \infty$, $p \neq 2$. A rich subspace X of $L_p(\mu)$ cannot be isometric to any $L_p(\nu)$-space (here ν need not be atomless). In particular, $L_p(\mu)$ contains no finite codimensional subspaces isometric to an $L_p(\nu)$-space.*

The second named author in [119] proved that no finite codimensional subspace is 1-complemented in any r.i. atomless Köthe function space on [0, 1], except for L_2. By Ando's theorem mentioned above, this is an exact extension of the last statement of Corollary 6.13.

Semenov asked whether Corollary 6.12 can be generalized to any r.i. Banach space on a finite atomless measure space.

Open problem 6.14. Let E be an r.i. space on a finite atomless measure space, $E \neq L_2$. Does there exist a constant $k_E > 1$ such that if $P \neq I$ is a projection onto a rich subspace of E then $\|P\| \geq k_E$?

In Section 6.3 we answer Open problem 6.14 affirmatively for Lorentz spaces on [0, 1] (Theorem 6.21).

An estimate of the best constant in the pseudo-Daugavet inequality in L_p

By the *best function* for the pseudo-Daugavet inequality we mean

$$\psi_p(t) = \inf\big\{\|I + T\| - 1 : T \in \mathcal{L}(L_p),\ T \text{ is narrow},\ \|T\| = t\big\} .$$

Boyko and V. Kadets [21] proved the following estimate of $\psi_p(t)$.

Theorem 6.15 (Boyko and V. Kadets [21]). *Let $1 < p < 2$ and $\psi_p : (0,\infty) \to (0,\infty)$ be the best function for the pseudo-Daugavet inequality. Then for every $t > 0$ we have*

$$\lim_{p \to 1+} \psi_p(t) = t .$$

For the proof we need some lemmas.

Lemma 6.16. *Given $1 \le p \le 2$, $\alpha > 0$, let $T_{p,\alpha} \in \mathcal{L}(L_p)$ be the operator defined for each $x \in L_p$ by*

$$T_{p,\alpha} x = -\alpha \int_{[0,1]} x \, \mathrm{d}\mu .$$

Then

$$\|I + T_{p,\alpha}\| \le (1+\alpha)^{\frac{2}{p}-1} \max\big\{1, |\alpha - 1|^{1-\frac{1}{p}}\big\} . \tag{6.19}$$

Proof. By the Daugavet property of L_1, $\|T_{1,\alpha}\| = 1 + \alpha$.

Let us calculate the norm $\|T_{2,\alpha}\|$. Since L_2 is a Hilbert space and $I + T_{2,\alpha}$ is a self-adjoint operator, we have

$$\begin{aligned} \|I + T_{2,\alpha}\| &= \sup_{x \in B_{L_2}} \Big| \int_{[0,1]} x \cdot (I + T_{2,\alpha})\, x \, \mathrm{d}\mu \Big| \\ &= \sup_{x \in B_{L_2}} \Big| \int_{[0,1]} \Big(x^2 - \alpha\, x \int_{[0,1]} x \, \mathrm{d}\mu \Big) \mathrm{d}\mu \Big| \\ &= \sup_{x \in B_{L_2}} \Big| 1 - \alpha \Big(\int_{[0,1]} x \, \mathrm{d}\mu \Big)^2 \Big| . \end{aligned}$$

Thus $\|I + T_{2,\alpha}\| = 1$, if $0 < \alpha \le 2$, and $\|I + T_{2,\alpha}\| = \alpha - 1$, if $2 < \alpha < \infty$. That is, $\|I + T_{2,\alpha}\| = \max\{1, \alpha - 1\}$.

By the Riesz–Thorin interpolation theorem [80, Theorem 2.b.14],

$$\|I + T_{p,\alpha}\| \le \|I + T_{1,\alpha}\|^{1-\theta} \|I + T_{2,\alpha}\|^{\theta} ,$$

where $\theta = 2 - \frac{2}{p}$. Plugging to the last inequality the obtained values of $\|I + T_{1,\alpha}\|$ and $\|I + T_{2,\alpha}\|$, we get (6.19). □

Corollary 6.17. *Let $1 < p < 2$. Then the best function for the pseudo-Daugavet inequality has the following estimate from above*

$$\psi_p(t) \le (1+t)^{\frac{2}{p}-1} \max\{1, |t-1|^{1-\frac{1}{p}}\}.$$

Recall that a function $f : (0, +\infty) \to \mathbb{R}$ is *convex* if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for all $x, y > 0$ and $\lambda \in [0, 1]$. It is well known that if f is twice differentiable at every point then f is convex if and only if $f''(t) \ge 0$ for all t.

Lemma 6.18. *For any $a > 0$ and $t > 0$, we set $F(t) = |a - t^{\frac{1}{p}}|^p$. Then the function $F : (0, +\infty) \to \mathbb{R}$ is convex. Moreover, $F''(t) > 0$ for each $t \in (0, a^p) \cup (a^p, +\infty)$.*

We omit the proof which is standard.

Given any $A \in \Sigma^+$, $t > 0$ and $d \ge 1$, we set

$$\varphi_d(t) = \min_{\varepsilon = \pm 1} \sup \left\{ \frac{\|\mathbf{1}_A + \varepsilon t^{\frac{1}{p}} \mathbf{1} + y\|^p}{\|\mathbf{1}_A + y\|^p} : y \in L_p^0(A), \|\mathbf{1}_A + y\| = d \|\mathbf{1}_A\| \right\},$$

where $\mathbf{1} = \mathbf{1}_{[0,1]}$. Observe that the values of φ_d do not depend on A. Let $\widetilde{\varphi}_d$ denote the lower convex envelope of the function φ_d, that is, the convex function defined on $[0, +\infty)$ such that if a convex function $\phi : [0, +\infty) \to \mathbb{R}$ satisfies $\phi(t) \le \varphi_d(t)$ for every $t \in [0, +\infty)$, then $\phi(t) \le \widetilde{\varphi}_d(t)$ for every $t \in [0, +\infty)$. One can show that

$$\widetilde{\varphi}_d(t) = \inf_{0 \le \alpha \le t < \beta} \left\{ \varphi_d(\alpha) + \Big(\varphi_d(\beta) - \varphi_d(\alpha)\Big) \frac{t - \alpha}{\beta - \alpha} \right\},$$

for each $t \in (0, +\infty)$. We also consider the function

$$\varphi_{d,+}(t) = \sup \left\{ \frac{\|\mathbf{1}_A - t^{\frac{1}{p}} \mathbf{1} + y\|^p}{\|\mathbf{1}_A + y\|^p} : y \in L_p^0(A), \|\mathbf{1}_A + y\| = d \|\mathbf{1}_A\|, \mathbf{1}_A + y \ge 0 \right\}.$$

We note that $\varphi_{d,+}(t)$ does not depend on the choice of $A \in \Sigma^+$.

Lemma 6.19. *For any $d > 1$ the function $\varphi_{d,+}$ is convex on $(0, +\infty)$ and thus*

$$\widetilde{\varphi}_d(t) \ge \varphi_{d,+}(t) \ \text{ for every } \ t \in (0, +\infty). \tag{6.20}$$

Proof. By Lemma 6.18, the function $|1 - t^{\frac{1}{p}} + y(t)|^p$ is convex with respect to the variable t. Thus, so is the function

$$\|\mathbf{1}_A - t^{\frac{1}{p}} \mathbf{1} + y\|^p = \int_A \left|1 - t^{\frac{1}{p}} + y(t)\right|^p \mathrm{d}\mu(t),$$

and hence, the function $\varphi_{d,+}$. Thus (6.20) follows from the definition of $\widetilde{\varphi}_d(t)$. □

Lemma 6.20. *Let $1 < p < 2$, $d > 1$, and $T \in \mathcal{L}(L_p)$ be a narrow operator. Then*

$$\|I + T\|^p \geq \varphi_{d,+}(\|T\|^p) .$$

Proof. Let $\varepsilon > 0$, and $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$ be a simple function with $\|x\| = 1$ and $Tx \neq 0$, where $[0,1] = A_1 \sqcup \ldots \sqcup A_m$, $A_k \in \Sigma^+$. Let $\varepsilon_1 \in (0, \|Tx\|/2)$ be such that

$$p(d + \|T\| + \varepsilon_1)^{p-1} 2\varepsilon_1 \leq \varepsilon d^p , \tag{6.21}$$

and

$$\text{if } t, s \in \left[\frac{\|Tx\|}{2}, 2\|Tx\|\right] \text{ and } |t - s| < \varepsilon_1 \text{ then } \left|\varphi_{d,+}(t^p) - \varphi_{d,+}(s^p)\right| < \varepsilon. \tag{6.22}$$

By the density of the simple functions, we may choose (partitioning, if necessary) the sets $(A_k)_{k=1}^m$ in such a way that there exist scalars $(b_k)_{k=1}^m$ with

$$\left\|Tx - \sum_{k=1}^m b_k \mathbf{1}_{A_k}\right\| < \varepsilon_1 . \tag{6.23}$$

Set

$$\varepsilon_2 = \frac{\varepsilon_1}{(1 + d)\sum_{k=1}^m |a_k|} . \tag{6.24}$$

Fix any $k \in \{1, \ldots, m\}$ and $\varepsilon_2 > 0$, and choose, by Proposition 2.19, an atomless sub-σ-algebra $\Sigma_k \subseteq \Sigma(A_k)$ such that

$$\left\|T\big|_{L_p^0(A_k, \Sigma_k, \mu|_{\Sigma_k})}\right\| < \varepsilon_2 . \tag{6.25}$$

Choose $y_k \in L_p^0(A_k, \Sigma_k, \mu|_{\Sigma_k})$ with $\|\mathbf{1}_{A_k} + y_k\| = d\|\mathbf{1}_{A_k}\|$. By (6.25), $\|Ty_k\| < \varepsilon_2\|y_k\|$. Since

$$\|y_k\| \leq \|\mathbf{1}_{A_k} + y_k\| + \|\mathbf{1}_{A_k}\| = (1 + d)\|\mathbf{1}_{A_k}\| \leq 1 + d ,$$

we obtain, using (6.24), that

$$\|Ty_k\| < \varepsilon_2(1 + d) = \frac{\varepsilon_1}{\sum_{k=1}^m |a_k|} . \tag{6.26}$$

Let G be the set of all possible such sequences $(y_k)_{k=1}^m$. Then

$$\|I + T\|^p \geq \sup_{(y_k)_{k=1}^m \in G} \frac{\left\|(I + T)\left(x + \sum_{k=1}^m a_k y_k\right)\right\|^p}{\left\|x + \sum_{k=1}^m a_k y_k\right\|^p} . \tag{6.27}$$

Since $\|x\| = 1$ and $\operatorname{supp} y_k \subseteq A_k$ for $k = 1, \ldots, m$,

$$\left\|x + \sum_{k=1}^m a_k y_k\right\|^p = \left\|\sum_{k=1}^m a_k(\mathbf{1}_{A_k} + y_k)\right\|^p = d^p \sum_{k=1}^m |a_k|^p \|\mathbf{1}_{A_k}\|^p = d^p . \tag{6.28}$$

We denote the numerator of (6.27) by α and estimate

$$\begin{aligned}
\alpha &= \Big\| x + Tx + \sum_{k=1}^{m} a_k y_k + \sum_{k=1}^{m} a_k T y_k \Big\| \\
&\geq \Big\| x + \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} + \sum_{k=1}^{m} a_k y_k \Big\| - \Big\| Tx - \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} \Big\| - \sum_{k=1}^{m} |a_k| \|T y_k\| \\
&\overset{\text{by (6.23) and (6.26)}}{\geq} \Big\| x + \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} + \sum_{k=1}^{m} a_k y_k \Big\| - 2\varepsilon_1 \, . \qquad (6.29)
\end{aligned}$$

Set $\beta = \|x + \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} + \sum_{k=1}^{m} a_k y_k\|$ and estimate

$$\beta \leq \Big\| x + \sum_{k=1}^{m} a_k y_k \Big\| + \|Tx\| + \Big\| Tx - \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} \Big\| \leq d + \|T\| + \varepsilon_1 \, . \qquad (6.30)$$

Since, for $t > s$ we have $(t-s)^p \geq t^p - pt^{p-1}s$, by (6.29) we obtain

$$\begin{aligned}
\alpha^p &\geq (\beta - 2\varepsilon_1) \geq \beta^p - p\beta^{p-1} 2\varepsilon_1 \overset{\text{by (6.30)}}{\geq} \beta^p - p\,(d + \|T\| + \varepsilon_1)\, 2\,\varepsilon_1 \\
&\overset{\text{by (6.21)}}{\geq} \beta^p - \varepsilon d^p \, . \qquad (6.31)
\end{aligned}$$

Using (6.28) and (6.31) we continue the estimate (6.27)

$$\begin{aligned}
\|I + T\|^p &\geq \frac{1}{d^p} \sup_{(y_k)_{k=1}^m \in G} \Big\| x + \sum_{k=1}^{m} b_k \mathbf{1}_{A_k} + \sum_{k=1}^{m} a_k y_k \Big\|^p - \varepsilon \\
&= \frac{1}{d^p} \sup_{(y_k)_{k=1}^m \in G} \sum_{k=1}^{m} \big\| a_k \mathbf{1}_{A_k} + b_k \mathbf{1}_{A_k} + a_k y_k \big\|^p - \varepsilon \\
&= \sup_{(y_k)_{k=1}^m \in G} \sum_{k=1}^{m} |a_k|^p \big\| \mathbf{1}_{A_k} \big\|^p \frac{\big\| \mathbf{1}_{A_k} + \frac{b_k}{a_k} a_k \mathbf{1}_{A_k} + y_k \big\|^p}{d^p \big\| \mathbf{1}_{A_k} \big\|^p} - \varepsilon \\
&\overset{\text{by the definition of } \varphi_d}{\geq} \sum_{k=1}^{m} \varphi_d \left(\frac{|b_k|^p}{|a_k|^p} \right) |a_k|^p \|\mathbf{1}_{A_k}\|^p - \varepsilon \, . \qquad (6.32)
\end{aligned}$$

Since $\varphi_d \geq \widetilde{\varphi}_d$, by the definition of $\widetilde{\varphi}_d$ and since $\widetilde{\varphi}_d \geq \varphi_{d,+}$ by Lemma 6.19, we can continue the estimate (6.32)

$$\begin{aligned}
\|I + T\|^p &\geq \sum_{k=1}^{m} \varphi_{d,+} \Big(\frac{|b_k|^p}{|a_k|^p} \Big) |a_k|^p \big\| \mathbf{1}_{A_k} \big\|^p - \varepsilon \\
&\geq \varphi_{d,+} \Big(\sum_{k=1}^{m} |b_k|^p \big\| \mathbf{1}_{A_k} \big\|^p \Big) - \varepsilon \, . \qquad (6.33)
\end{aligned}$$

Observe that (6.23) implies $\big|\|Tx\| - \|\sum_{k=1}^m b_k \mathbf{1}_{A_k}\|\big| < \varepsilon_1$, and since $\varepsilon_1 < \|Tx\|/2$, by (6.22), we continue the estimate (6.33) as follows:

$$\|I + T\|^p \ge \varphi_{d,+}\left(\|Tx\|^p\right) - 2\varepsilon \, .$$

Taking the supremum over all simple functions x with $\|x\| = 1$ and $Tx \neq 0$, we get

$$\|I + T\|^p \ge \varphi_{d,+}\left(\|T\|^p\right) - 2\varepsilon \, .$$

By arbitrariness of $\varepsilon > 0$, the lemma is proved. □

Proof of Theorem 6.15. Let $t > 1$ and $T \in \mathcal{L}(L_p)$ be a narrow operator with $\|T\| = t$. By arbitrariness of $d > 1$ in Lemma 6.20,

$$\|I + T\|^p \ge \sup_{d>1} \varphi_{d,+}\left(\|T\|^p\right) \, ,$$

and thus,

$$\psi_p(t) \ge \Big(\sup_{d>1} \varphi_{d,+}(t^p)\Big)^{1/p} - 1 \, .$$

By the definition of $\varphi_{d,+}$ for $A = [0,1]$, making the substitution $x = \mathbf{1} + y$, we obtain

$$\varphi_{d,+}(t) = \sup\left\{\frac{\left\|x - t^{1/p}\mathbf{1}\right\|^p}{\|x\|^p} : x \in L_1^+, \ \|x\| = d, \ \int_{[0,1]} x \, \mathrm{d}\mu = 1\right\} .$$

Hence,

$$\sup_{d>1} \varphi_{d,+}(t) = \sup\left\{\frac{\left\|x - t^{1/p}\mathbf{1}\right\|^p}{\|x\|^p} : x \in L_1^+, \ \int_{[0,1]} x \, \mathrm{d}\mu = 1\right\} .$$

Applying the substitution $x = (\int_{[0,1]} y \, \mathrm{d}\mu)^{-1} y$, we obtain

$$\sup_{d>1} \varphi_{d,+}(t) = \sup\left\{\frac{\left\|y - t^{1/p}\left(\int_{[0,1]} y \, \mathrm{d}\mu\right)\mathbf{1}\right\|^p}{\|y\|^p} : y \in L_1^+, \ y \neq 0\right\} .$$

Let $s = (\frac{2}{p-1})^{\frac{1}{p-2}}$, and choose $A \in \Sigma$ with $\mu(A) = s$ (this is possible since $0 < s < 1$). Let $x_0 = s^{-1/p}\mathbf{1}_A$. Since $x_0 \ge 0$ and $\|x_0\| = 1$, we can estimate

$$\begin{aligned}
\psi_p(t) &\ge \Big(\sup_{d>1} \varphi_{d,+}\left(t^p\right)\Big)^{1/p} - 1 \ge \Big\|x_0 - t\int_{[0,1]} x_0 \, \mathrm{d}\mu\Big\| - 1 \\
&= \Big(\Big|s^{-\frac{1}{p}} - t s^{1-\frac{1}{p}}\Big|^p s + t^p s^{p-1}(1-s)\Big)^{1/p} - 1 \\
&= \Big(|1 - ts|^p + t^p s^{p-1} - t^p s^p\Big)^{1/p} - 1 \\
&= \left(\left|1 - t\Big(\frac{2}{p-1}\Big)^{\frac{1}{p-2}}\right|^p + t^p\Big(\frac{2}{p-1}\Big)^{\frac{p-1}{p-2}} - t^p\Big(\frac{2}{p-1}\Big)^{\frac{p}{p-2}}\right)^{1/p} - 1 \, .
\end{aligned}$$

We have

$$\lim_{p\to 1+}\left(\frac{2}{p-1}\right)^{\frac{1}{p-2}} = 0, \quad \lim_{p\to 1+}\left(\frac{2}{p-1}\right)^{\frac{p-1}{p-2}} = 1, \quad \lim_{p\to 1+}\left(\frac{2}{p-1}\right)^{\frac{p}{p-2}} = 0,$$

and hence,

$$\lim_{p\to 1+} \psi_p(t) \geq 1 + t - 1 = t\,.$$

On the other hand, the inequality $\psi_p(t) \leq t$ for each $t > 0$ and $p \geq 1$ follows from the inequality $\|I + T\| \leq 1 + \|T\|$. □

6.3 A pseudo-Daugavet property for narrow projections in Lorentz spaces

Lorentz spaces were introduced by Lorentz [82, 83] in connection with some problems of harmonic analysis and interpolation theory. Since then they have become one of the most important examples of r.i. spaces and were extensively studied by many authors. We start by recalling the definition of Lorentz spaces and introducing some notation.

If f is a measurable function, we define the *nonincreasing rearrangement* of f to be

$$f^*(t) = \inf\{s : \mu(|f| > s) \leq t\}\,.$$

Notice that when f is a simple function, $f = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k}$, then f^* is also a simple function and the range of f^* equals $\{|a_k| : k = 1, \ldots, m\}$.

If $1 \leq p < \infty$ and if $w : (0,1) \to (0,\infty)$ is a nonincreasing function, we define the *Lorentz norm* of a measurable function f to be

$$\|f\|_{w,p} = \left(\int_{[0,1]} w(t) f^*(t)^p \, dt\right)^{1/p}.$$

The *Lorentz space* $L_{w,p}([0,1],\mu)$ is the space of those measurable functions f for which $\|f\|_{w,p}$ is finite. These spaces are a generalization of the L_p spaces: if $w(x) = 1$ for all $0 \leq x < 1$, then $L_{w,p} = L_p$ with equality of norms.

The main result of this section answers the Open problem 6.14 of Semenov for Lorentz spaces on $[0,1]$.

Theorem 6.21. *Suppose $L_{p,w}$ is a Lorentz space on $[0,1]$ and $p > 2$. Then there exists $\varrho_p > 1$ such that for every nontrivial projection P from $L_{p,w}$ onto a rich subspace*

$$\|P\| \geq \varrho_p\,.$$

In the proof of Theorem 6.21 we will use the following two propositions.

Proposition 6.22. *Suppose $L_{p,w}$ is a Lorentz space on $[0,1]$ with $p > 2$. Then there exist $\delta_p \in (0, 1/8)$, $\lambda_p = \lambda_p(\delta_p, p) \in (\delta_p/(\delta_p - 4), 0)$ and $\gamma_p = \gamma_p(\lambda_p, \delta_p, p) \in (0,1)$ such that*

$$\gamma_p + 2|\lambda_p|\delta_p < 1$$

which satisfy the following property:

for every simple function $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$ such that $1 \le \|x\|_{p,w} \le 1 + \frac{3}{2}\delta_p$ and

$$\frac{|a_i|}{|a_j|} \notin (3 - \delta_p, 3)$$

for all $i, j = 1, \dots, m$; and for every partition $A_k = B_k \sqcup C_k$ with $\mu(B_k) = (1/4)\mu(A_k)$ we have

$$\Big\| \sum_{k=1}^m a_k (3\mathbf{1}_{B_k} - \mathbf{1}_{C_k}) \Big\|_{p,w} \ge \left(\frac{3}{4}\right)^{\frac{1}{p}} ,$$

and

$$\Big\| \lambda_p x + \sum_{k=1}^m a_k (3\mathbf{1}_{B_k} - \mathbf{1}_{C_k}) \Big\|_{p,w} \le \gamma_p \Big\| \sum_{k=1}^m a_k (3\mathbf{1}_{B_k} - \mathbf{1}_{C_k}) \Big\|_{p,w} .$$

Proposition 6.23. *Let X be an r.i. space. Given a simple function $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k} \in X$ and $\delta \in (0, 1/8)$, there exists a simple function $x^\sharp = x^\sharp(\delta) = \sum_{k=1}^m a_k^\sharp \mathbf{1}_{A_k}$ such that for all $i, j = 1, \dots, m$,*

$$\frac{|a_i^\sharp|}{|a_j^\sharp|} \notin (3 - \delta, 3)$$

and $\|x - x^\sharp\| < (3/2)\delta$, $\|x\| \le \|x^\sharp\| < (1 + (3/2)\delta)\|x\|$.

Let us first show that Theorem 6.21 is indeed a consequence of Propositions 6.22 and 6.23.

Proof of Theorem 6.21. Fix $\varepsilon > 0$. Since P is a nontrivial projection, there exists a simple function $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$, $a_k \neq 0$, with $\|x\| = 1$ and $\|Px\| < \varepsilon$. Note that since we will always work in $L_{p,w}$ we will drop the subscript and simply use $\|\cdot\|$ to mean $\|\cdot\|_{p,w}$ throughout this proof.

Let δ_p be as defined in the statement of Proposition 6.22. Since $\delta_p \in (0, 1/8)$, by Proposition 6.23, there exists a simple function $x^\sharp = \sum_{k=1}^m a_k^\sharp \mathbf{1}_{A_k}$ with $1 \le \|x^\sharp\| < 1 + (3/2)\delta_p$, $\|x - x^\sharp\| < (3/2)\delta_p$ and such that

$$\frac{|a_i^\sharp|}{|a_j^\sharp|} \notin (3 - \delta_p, \delta_p) , \tag{6.34}$$

for all $i, j = 1, \dots, m$.

Since $I - P$ is narrow, by Lemma 1.11, for each k, $1 \le k \le m$, there exists a partition $A_k = B_k \sqcup C_k$ such that $\mu(B_k) = (1/4)\mu(A_k)$ and

$$\|(I-P)(3\mathbf{1}_{B_k} - \mathbf{1}_{C_k})\| < \frac{\varepsilon}{|a_k^\sharp| m} .$$

Then for $z = \sum_{k=1}^m a_k^\sharp (3\mathbf{1}_{B_k} - \mathbf{1}_{C_k})$ we obtain $\|(I-P)\, z\| < \varepsilon$.

Moreover, by (6.34) and Proposition 6.22 we conclude that

$$\|\lambda_p x^\sharp + z\| \le \gamma_p \|z\| ,$$

where λ_p and γ_p are constants defined in Proposition 6.22. By Proposition 6.22, $\|z\|^p \ge 3/4$ and thus,

$$\begin{aligned}
\|z\| &= \|P(\lambda_p x^\sharp + z) - P\lambda_p x^\sharp - P\lambda_p x + P\lambda_p x + z - Pz\| \\
&\le \|P\| \cdot \|\lambda_p x^\sharp + z\| + |\lambda_p| \cdot \|P\| \cdot \|x^\sharp - x\| + |\lambda_p| \|Px\| + \|(I-P)z\| \\
&\le \|P\| \cdot \gamma_p \|z\| + \|P\| |\lambda_p| \cdot \frac{3}{2}\delta_p + |\lambda_p|\varepsilon + \varepsilon \\
&\le \|z\| \cdot \|P\| \Big(\gamma_p + \frac{3}{2}\delta_p |\lambda_p| \Big(\frac{4}{3}\Big)^{\frac{1}{p}} \Big) + \varepsilon(|\lambda_p| + 1) \\
&\le \|z\| \cdot \|P\| (\gamma_p + 2\delta_p |\lambda_p|) + \varepsilon(|\lambda_p| + 1).
\end{aligned}$$

Since ε was arbitrary, we obtain

$$\|P\| \ge (\gamma_p + 2\delta_p|\lambda_p|)^{-1} \stackrel{\text{def}}{=} \varrho_p .$$

By Proposition 6.22, $\varrho_p > 1$. □

Remark 6.24. Note that the same proof will demonstrate that whenever T is a narrow operator on $L_{p,w}$, $p > 2$, such that 1 is an eigenvalue of T, i.e. such that there exists a nonzero element $x \in L_{p,w}$ with $Tx = x$, then

$$\|I - T\| \ge \varrho_p > 1 .$$

(Simply replace P in the proof with $I - T$, and note that Propositions 6.22 and 6.23 do not depend on the operator at all.)

Proof of Proposition 6.22. Let $\delta \in (0, 1/8)$ and $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$ be a simple function such that $1 \le \|x\|_{p,w} \le 1 + (3/2)\delta$ and

$$\frac{|a_i|}{|a_j|} \notin (3 - \delta, 3) ,$$

for all $i, j = 1, \ldots, m$. We assume without loss of generality that $|a_1| \geq |a_2| \geq \ldots \geq |a_m|$. For $k = 1, \ldots, m$, let B_k, C_k be subsets so that $A_k = B_k \sqcup C_k$ and $\mu(B_k) = (1/4)\mu(A_k)$. We denote

$$y = \sum_{k=1}^{m} 3a_k \mathbf{1}_{B_k} - a_k \mathbf{1}_{C_k}, \ \alpha_k = \sum_{j=1}^{k} \mu(A_j)$$

and

$$\gamma_k = \sum_{j=1}^{k} \mu(C_j) \text{ for } k = 1, \ldots, m \,.$$

First note that

$$1 \leq \|x\|^p = \sum_{k=1}^{m} |a_k| \int_{\alpha_{k-1}}^{\alpha_k} w(t)dt,$$

and

$$\|y\|^p \geq \| \sum_{k=1}^{m} |a_k| \mathbf{1}_{C_k} \|^p = \sum_{k=1}^{m} |a_k| \int_{\gamma_{k-1}}^{\gamma_k} w(t)dt.$$

Since $w(t)$ is a nonnegative, nonincreasing function, we have that

$$\begin{aligned}
\int_{\alpha_{k-1}}^{\alpha_k} w(t)dt &= \int_{\alpha_{k-1}}^{\alpha_{k-1}+\frac{3}{4}\mu(A_k)} w(t)dt + \int_{\alpha_{k-1}+\frac{3}{4}\mu(A_k)}^{\alpha_k} w(t)dt \\
&\leq \frac{4}{3} \int_{\alpha_{k-1}}^{\alpha_{k-1}+\frac{3}{4}\mu(A_k)} w(t)dt,
\end{aligned}$$

and

$$\int_{\gamma_{k-1}}^{\gamma_k} w(t)dt = \int_{\frac{3}{4}\alpha_{k-1}}^{\frac{3}{4}\alpha_k} w(t)dt \geq \int_{\alpha_{k-1}}^{\alpha_{k-1}+\frac{3}{4}\mu(A_k)} w(t)dt.$$

Thus

$$\begin{aligned}
\|y\|^p &\geq \sum_{k=1}^{m} |a_k| \int_{\gamma_{k-1}}^{\gamma_k} w(t)dt \geq \sum_{k=1}^{m} |a_k| \int_{\alpha_{k-1}}^{\alpha_{k-1}+\frac{3}{4}\mu(A_k)} w(t)dt \\
&\geq \sum_{k=1}^{m} |a_k| \frac{3}{4} \int_{\alpha_{k-1}}^{\alpha_k} w(t)dt = \frac{3}{4}\|x\|^p \geq \frac{3}{4},
\end{aligned}$$

which proves the first part of the conclusion.

For the second part, we set $b_k = 3a_k, c_k = -a_k$ for $k = 1, \ldots, m$.

Thus,

$$y = \sum_{k=1}^{m} b_k \mathbf{1}_{B_k} + c_k \mathbf{1}_{C_k},$$

$$\lambda x + y = \sum_{k=1}^{m} b_k \big(1 + \frac{\lambda}{3}\big) \mathbf{1}_{B_k} + c_k (1-\lambda) \mathbf{1}_{C_k}.$$

We first notice that if $0 \geq \lambda > \delta/(\delta - 4) > -1$ then for all $i, j = 1, \ldots, m$ the following hold:

$$|b_i|\Big(1 + \frac{\lambda}{3}\Big) \leq |b_j|\Big(1 + \frac{\lambda}{3}\Big) \Longleftrightarrow |b_i| \leq |b_j|; \tag{6.35}$$

$$|c_i|(1-\lambda) \leq |c_j|(1-\lambda) \Longleftrightarrow |c_i| \leq |c_j|; \tag{6.36}$$

$$|c_i|(1-\lambda) < |b_i|\Big(1 + \frac{\lambda}{3}\Big) \quad \text{for all } i; \tag{6.37}$$

$$|b_i|\Big(1 + \frac{\lambda}{3}\Big) \leq |c_j|(1-\lambda) \Longleftrightarrow |b_i| \leq |c_j|; \tag{6.38}$$

$$|b_i|\Big(1 + \frac{\lambda}{3}\Big) \neq |c_j|(1-\lambda) \quad \text{for all } \ i, j. \tag{6.39}$$

Indeed (6.35), (6.36) and (6.37) are obvious since $1 + \lambda/3 > 0$, $1 - \lambda > 0$ and $1 - \lambda < 3(1 + \lambda/3)$. To see (6.38)"$\Rightarrow$," suppose on the contrary, that there exist i, j such that

$$|b_i|(1 + \frac{\lambda}{3}) \leq |c_j|(1-\lambda), \ \text{ and } \ |b_i| > |c_j| \, .$$

Then

$$1 > \frac{|c_j|}{|b_i|} \geq \frac{1-\lambda}{1 + \frac{\lambda}{3}} = \frac{1-\lambda}{3(3+\lambda)} \, .$$

Thus, since $\lambda \in (\delta/(\delta-4), 0)$,

$$3 > \frac{|a_j|}{|a_i|} \geq \frac{1-\lambda}{3+\lambda} > \frac{3 - \frac{\delta}{4-\delta}}{1 + \frac{\delta}{4-\delta}} = 3 - \delta \, ,$$

which contradicts (6.37), and (6.38)"$\Rightarrow$" is proved.

Next, suppose $|b_i| \leq |c_j|$. Since $\lambda < 0$ we get

$$|b_i|(1 + \frac{\lambda}{3}) < |b_i| \leq |c_j| < |c_j|(1-\lambda) \, .$$

Thus (6.38)"$\Leftarrow$" and (6.39) are proved.

Now define numbers t_{C_k}, t_{B_k} for $k = 1, \ldots, m$ as follows:

$$t_{C_i} = \sum_{k<i} \mu(C_k) + \sum_{l:|b_l|>|c_i|} \mu(B_l),$$

$$t_{B_j} = \sum_{k<j} \mu(B_k) + \sum_{i:|c_i|\geq|b_j|} \mu(C_i).$$

It follows from (6.35)–(6.39) that for all $i, j = 1, \dots, m$

$$
\begin{aligned}
t_{C_i} > t_{C_j} \quad &\Rightarrow \quad \Big(|c_i| \le |c_j| \ \text{ and } \ |c_i|(1-\lambda) \le |c_j|(1-\lambda)\Big),\\
\Big(|c_i| < |c_j|, \ \text{or, equivalently, } \ |c_i|(1-\lambda) < |c_j|(1-\lambda)\Big) \quad &\Rightarrow \quad t_{C_i} > t_{C_j},\\
t_{B_i} > t_{B_j} \quad &\Rightarrow \quad \Big(|b_i| \le |b_j| \ \text{ and } \ |b_i|(1+\frac{\lambda}{3}) \le |b_j|(1+\frac{\lambda}{3})\Big),\\
\Big(|b_i| < |b_j| \ \text{ or, equivalently, } \ |b_i|(1+\frac{\lambda}{3}) < |b_j|(1+\frac{\lambda}{3})\Big) \quad &\Rightarrow \quad t_{B_i} > t_{B_j},\\
t_{B_i} > t_{C_i} \quad \Leftrightarrow \quad |b_i|(1+\frac{\lambda}{3}) < |c_j|(1-\lambda) \quad &\Leftrightarrow \quad |b_i| \le |c_j|.
\end{aligned}
\tag{6.40}
$$

Now define the following weights:

$$
w_{B_k} = \int_{t_{B_k}}^{t_{B_k}+\mu(B_k)} w \, \mathrm{d}\mu \ , \qquad w_{C_k} = \int_{t_{C_k}}^{t_{C_k}+\mu(C_k)} w \, \mathrm{d}\mu \ .
$$

By (6.40) we obtain

$$
\begin{aligned}
\|y\|_{p,w}^p &= \sum_{k=1}^m \big[|b_k|^p w_{B_k} + |c_k|^p w_{C_k}\big],\\
\|\lambda x + y\|_{p,w}^p &= \sum_{k=1}^m \big[|b_k|^p (1+\frac{\lambda}{3})^p w_{B_k} + |c_k|^p (1-\lambda)^p w_{C_k}\big].
\end{aligned}
$$

Thus, the nonincreasing order of moduli of coefficients of y is the same as the nonincreasing order of moduli of coefficients of $\lambda x + y$.

Thus if we set

$$
\psi(\lambda) = \sum_{k=1}^m \big[|b_k|^p (1+\frac{\lambda}{3})^p w_{B_k} + |c_k|^p (1-\lambda)^p w_{C_k}\big] ,
$$

for $\lambda \in (-3, 1)$, then

$$
\begin{aligned}
\psi(0) &= \|y\|_{p,w}^p \ge \frac{3}{4},\\
\psi(\lambda) &= \|\lambda x + y\|_{p,w}^p \ \text{ for } \ \lambda \in (\delta/(\delta-4), 0).
\end{aligned}
\tag{6.41}
$$

Clearly, ψ is differentiable for all $\lambda \in (-3, 1)$ and

$$
\psi'(\lambda) = \sum_{k=1}^m \big[p|b_k|^p (1+\frac{\lambda}{3})^{p-1} \frac{1}{3} w_{B_k} - p|c_k|^p (1-\lambda)^{p-1} w_{C_k}\big] .
$$

Thus,

$$\psi'(0) = p\sum_{k=1}^{m}\Big[\frac{1}{3}|b_k|^p w_{B_k} - |c_k|^p w_{C_k}\Big] = p\sum_{k=1}^{m}|a_k|^p\big[3^{p-1}w_{B_k} - w_{C_k}\big]. \quad (6.42)$$

We now need to compare the quantities w_{B_k} and w_{C_k} for a given k, $1 \le k \le m$. It follows from (6.37) that $t_{C_k} > t_{B_k}$. Moreover, by definition of B_k and C_k, we have $\mu(C_k) = 3\mu(B_k)$. Thus, since w is nonincreasing, we obtain

$$\begin{aligned} w_{C_k} &= \int_{t_{C_k}}^{t_{C_k}+\mu(C_k)} w\,\mathrm{d}\mu = \int_{t_{C_k}}^{t_{C_k}+3\mu(B_k)} w\,\mathrm{d}\mu \\ &\le \int_{t_{B_k}}^{t_{B_k}+\mu(B_k)} w\,\mathrm{d}\mu + \int_{t_{B_k}+\mu(B_k)}^{t_{B_k}+2\mu(B_k)} w\,\mathrm{d}\mu + \int_{t_{B_k}+2\mu(B_k)}^{t_{B_k}+3\mu(B_k)} w\,\mathrm{d}\mu \\ &\le 3\int_{t_{B_k}}^{t_{B_k}+\mu(B_k)} w\,\mathrm{d}\mu = 3w_{B_k}. \end{aligned}$$

Thus for all $k = 1, \dots, m$,

$$3^{p-1}w_{B_k} - w_{C_k} \ge 3^{p-1}w_{B_k} - 3w_{B_k} = w_{B_k}(3^{p-1} - 3)\,. \quad (6.43)$$

In analogy to numbers $t_{C_k}, t_{B_k}, w_{C_k}, w_{B_k}$ we define

$$t_{A_k} = \sum_{l<k}\mu(A_l), \quad w_{A_k} = \int_{t_{A_k}}^{t_{A_k}+\mu(A_k)} w d\mu\,.$$

Since we assumed that $|a_1| \ge |a_2| \ge \ldots \ge |a_m|$, we obtain

$$\|x\|_{p,w}^p = \sum_{k=1}^{m}|a_k|^p w_{A_k}\,.$$

Further, since for all k, $\mu(A_k) = \mu(B_k) + \mu(C_k)$ and since $|c_l| \ge |b_j| \Rightarrow |c_l| > |c_j| \Rightarrow l < j$, we obtain

$$\begin{aligned} t_{B_j} &= \sum_{k<j}\mu(B_k) + \sum_{l:|c_l|\ge|b_j|}\mu(C_l) \le \sum_{k<j}\mu(B_k) + \sum_{l<j}\mu(C_l) \\ &= \sum_{k<j}\mu(A_k) = t_{A_j}\,. \end{aligned}$$

Therefore, since for all $j = 1, \dots, m$, $\mu(B_j) = \frac{1}{4}\mu(A_j)$, we obtain

$$w_{B_j} = \int_{t_{B_j}}^{t_{B_j}+\mu(B_j)} w\,\mathrm{d}\mu \ge \int_{t_{A_j}}^{t_{A_j}+\frac{1}{4}\mu(A_j)} w\,\mathrm{d}\mu \ge \frac{1}{4}\int_{t_{A_j}}^{t_{A_j}+\mu(A_j)} w\,\mathrm{d}\mu = \frac{1}{4}w_{A_j}\,.$$

Thus, we can continue the estimate from (6.43) as follows:

$$3^{p-1} w_{B_k} - w_{C_k} \geq w_{B_k}(3^{p-1} - 3) \geq \frac{1}{4}(3^{p-1} - 3) w_{A_k} \,.$$

Plugging this into (6.42), we get

$$\begin{aligned} \psi'(0) &= p \sum_{k=1}^{m} |a_k|^p \big[3^{p-1} w_{B_k} - w_{C_k}\big] \geq \frac{1}{4} p(3^{p-1} - 3) \sum_{k=1}^{m} |a_k|^p w_{A_k} \\ &= \frac{1}{4} p(3^{p-1} - 3) \|x\|_{w,p}^p \geq \frac{1}{4} p(3^{p-1} - 3) \overset{\text{def}}{=} C_p. \end{aligned} \tag{6.44}$$

Note that our assumption that $p > 2$ guarantees that $C_p > 0$.

Our next step is to estimate from above the value of $|\psi''(\lambda)|$ when $\lambda \in (\delta/(\delta - 4), \delta/(4 - \delta))$. We have, since $\delta \in (0, \frac{1}{8})$,

$$\begin{aligned} |\psi''(\lambda)| &= \Big| p(p-1) \sum_{k=1}^{m} \Big[\frac{1}{9}|b_k|^p (1 + \frac{\lambda}{3})^{p-2} w_{B_k} + |c_k|^p (1 - \lambda)^{p-2} w_{C_k}\Big]\Big| \\ &\leq p(p-1)\Big(1 + \frac{\delta}{4 - \delta}\Big)^{p-2} \sum_{k=1}^{m} \Big[\frac{1}{9}|b_k|^p w_{B_k} + |c_k|^p w_{C_k}\Big] \\ &\leq p(p-1)\Big(1 + \frac{\delta}{4 - \delta}\Big)^{p-2} \|y\|_{w,p}^p \leq p(p-1)\Big(\frac{4}{4 - \delta}\Big)^{p-2} 3^p \|x\|_{w,p}^p \\ &\leq p(p-1)\Big(\frac{4}{4 - \delta}\Big)^{p-2} 3^p (1 + \frac{3}{2}\delta)^p \leq p(p-1) 4^p \overset{\text{def}}{=} M_p. \end{aligned} \tag{6.45}$$

By Taylor's Theorem for $\lambda \in (\delta/(\delta - 4), 0)$ we get

$$\psi(\lambda) = \psi(0) + \lambda\psi'(0) + \frac{1}{2}\lambda^2 \psi''(\theta) \,,$$

where $\theta \in (\lambda, 0)$.

Thus, by (6.44), (6.45) and since $\psi(0) = \|y\|^p \geq 3/4$, we have

$$\psi(\lambda) \leq \psi(0) + \lambda C_p + \frac{1}{2}\lambda^2 M_p \leq \psi(0)[1 + \frac{4}{3}\lambda(C_p + \frac{1}{2}\lambda M_p)] \,.$$

Thus, when $\lambda < 0$ and $|\lambda| \leq \min\{\delta/(4 - \delta), (C_p/M_p)\}$, we get

$$\psi(\lambda) \leq \psi(0)\big[1 + \frac{2}{3}\lambda C_p\big] \,.$$

If $\delta \leq \min\{1/8, 4C_p/M_p, (2C_p)/(3p)\}$, we set $\lambda = -\delta/4$ and

$$\gamma(\delta) \overset{\text{def}}{=} (1 - \frac{2}{3}\delta C_p)^{\frac{1}{p}} \,.$$

Then $\gamma(\delta) < 1$ and by (6.41), we have

$$\left\|\lambda x + y\right\| \le \gamma(\delta)\|y\| \,.$$

Further, by the Bernoulli inequality, $\gamma(\delta) < 1 - \frac{2}{3}\delta\frac{C_p}{p}$. Hence

$$\gamma(\delta) + 2|\lambda|\delta < 1 - \frac{2}{3}\delta\frac{C_p}{p} + \frac{1}{2}\delta\frac{C_p}{p} < 1,$$

and the proposition is proved. □

Remark 6.25. The above proof does not work for $p < 2$. Indeed, when $p < 2$, the estimate (6.43) becomes meaningless and both constants C_p and D_p are negative. Moreover, for every p, $1 \le p < 2$, it is not difficult to construct weights w_p such that when $x = \mathbf{1}_{[0,1]}$ is partitioned into any disjoint sets $[0,1] = B \sqcup C$ with $\mu(B) = 1/4$ then for any $\lambda \in \mathbb{R}$

$$\|\lambda x + (3\mathbf{1}_B - \mathbf{1}_C)\|_{p,w_p} \ge \|3\mathbf{1}_B - \mathbf{1}_C\|_{p,w_p} \,.$$

In fact, one can take, e.g.

$$w_p = \frac{4}{3^{p-1}+1}\mathbf{1}_{[0,\frac{1}{4})} + \frac{4\cdot 3^{p-2}}{3^{p-1}+1}\mathbf{1}_{[\frac{1}{4},1]} \,.$$

This is a well-defined weight when $1 \le p < 2$. It is routine, even though tedious, to check that L_{p,w_p} satisfy (6.42) for all p with $1 \le p < 2$. We leave the details to the interested reader.

Proof of Proposition 6.23

The first step of the proof of Proposition 6.23 is the following lemma.

Lemma 6.26. *Let X be an r.i. space and x be a simple function, $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$. For any $\eta > 0$ there exists $\overline{x} = \overline{x}(\eta) = \sum_{k=1}^m \overline{a}_k \mathbf{1}_{A_k}$ such that for all $i, j = 1, \ldots, m$,*

$$\frac{|\overline{a}_j|}{|\overline{a}_i|} \notin (1, 1+\eta)$$

and $\|x - \overline{x}\| < \eta\|x\|$, $\|\overline{x}\| \ge \|x\|$.

Proof of Lemma 6.26. Without loss of generality we assume that $|a_1| \ge |a_2| \ge \ldots \ge |a_m|$. Let $r_0 = 1 < r_1 < r_2 \ldots < r_n = m$ be such that

$$\frac{|a_j|}{|a_i|} \begin{cases} < 1+\eta & \text{if there exists } k \text{ with } r_k \le j < i < r_{k+1}, \\ \ge 1+\eta & \text{if there exists } k \text{ with } j < r_k \le i. \end{cases}$$

Define $\overline{a}_j = \operatorname{sgn}(a_j)|a_{r_{k(j)}}|$, where $k(j)$ is such that $r_{k(j)} \leq j < r_{k(j)+1}$. Then for any $j < i$ we have $k(j) \leq k(i)$ and

$$\frac{|\overline{a}_j|}{|\overline{a}_i|} = \frac{|a_{r_{k(j)}}|}{|a_{r_{k(i)}}|} \begin{cases} = 1 & \text{if } k(j) = k(i), \\ \geq 1 + \eta & \text{if } k(j) < k(i). \end{cases}$$

Thus

$$\frac{|\overline{a}_j|}{|\overline{a}_i|} \notin (1, 1 + \eta)$$

as required.

Moreover for all $j = 1, \ldots, m$

$$\frac{\overline{a}_j}{a_j} = \frac{|a_{r_{k(j)}}|}{|a_j|} \in [1, 1 + \eta) \ .$$

Thus $\|\overline{x}\| \geq \|x\|$ and

$$\begin{aligned} \|\overline{x} - x\| &= \Big\| \sum_{j=1}^{m} (\overline{a}_j - a_j) \mathbf{1}_{A_k} \Big\| = \Big\| \sum_{j=1}^{m} a_j \big(\frac{\overline{a}_j}{a_j} - 1\big) \mathbf{1}_{A_k} \Big\| \\ &< \eta \Big\| \sum_{j=1}^{m} a_j \mathbf{1}_{A_j} \Big\| = \eta \|x\|. \end{aligned}$$

□

In the next lemma we gather, for easy reference, a few simple arithmetic inequalities which will be useful in the proof of Proposition 6.23.

Lemma 6.27. *Let $\delta \in (0, 3)$ and $\tilde{t_i}, t_i$, $i = 1, 2, 3, 4$, be positive real numbers such that*

$$\frac{\tilde{t_i}}{t_i} \in \Big[1, \frac{3}{3 - \delta}\Big) \ \ \text{for } i = 1, 2, 3, 4 \ .$$

Then we have:

(i) *If $t_i = t_2$, $\tilde{t_i}/\tilde{t_3} \in (3 - \delta, 3)$, $\tilde{t_2}/\tilde{t_4} \in (3 - \delta, 3)$, then $\tilde{t_3}/\tilde{t_4} \in ((3 - \delta)^3/27, 27/(3 - \delta)^3)$.*

(ii) *If $t_1/t_2 \in (3 - \delta, 3)$, $t_3/t_2 = 3$, then $t_3/t_1 \in (1, 3/(3 - \delta))$.*

(iii) *If $t_1 < t_2$, then $\tilde{t_1}/\tilde{t_2} < 9/(3 - \delta)^2$.*

(iv) *If $t_1/\tilde{t_2} = 3$, $t_1/\tilde{t_3} \in (3 - \delta, 3)$, then $t_2/t_3 \in ((3 - \delta)^2/9, 3/(3 - \delta))$.*

Proof of Lemma 6.27. The proofs of Lemma 6.27(i)–(iv) are very simple and very similar to each other. As an illustration, we prove implication (i).

We have

$$\frac{t_3}{t_4} = \frac{t_3}{\tilde{t_3}} \cdot \frac{\tilde{t_3}}{\tilde{t_1}} \cdot \frac{\tilde{t_1}}{t_1} \cdot \frac{t_1}{t_2} \cdot \frac{t_2}{\tilde{t_2}} \cdot \frac{\tilde{t_2}}{\tilde{t_4}} \cdot \frac{\tilde{t_4}}{t_4} \,.$$

Since

$$\frac{t_3}{\tilde{t_3}} \in \Big(\frac{3-\delta}{3}, 1\Big], \ \frac{\tilde{t_3}}{\tilde{t_1}} \in \Big(\frac{1}{3}, \frac{1}{3-\delta}\Big), \ \frac{\tilde{t_1}}{t_1} \in \Big[1, \frac{3}{3-\delta}\Big), \ \frac{t_1}{t_2} = 1,$$
$$\frac{t_2}{\tilde{t_2}} \in \Big(\frac{3-\delta}{3}, 1\Big], \ \frac{\tilde{t_2}}{\tilde{t_4}} \in (3-\delta, 3), \ \frac{\tilde{t_4}}{t_4} \in \Big[1, \frac{3}{3-\delta}\Big),$$

we obtain

$$\frac{t_3}{t_4} \in \Big(\frac{3-\delta}{3} \cdot \frac{1}{3} \cdot 1 \cdot 1 \cdot \frac{3-\delta}{3} \cdot (3-\delta) \cdot 1, 1 \cdot \frac{1}{3-\delta} \cdot \frac{3}{3-\delta} \cdot 1 \cdot 1 \cdot 3 \cdot \frac{3}{3-\delta}\Big)$$
$$= \Big(\frac{(3-\delta)^3}{27}, \frac{27}{(3-\delta)^3}\Big).$$

Implications (ii)–(iv) are proved in a very similar way. □

For any simple function $y = \sum_{k=1}^m d_k \mathbf{1}_{D_k}$ we define the sets

$$S(y,k) \overset{\text{def}}{=} \{j \in \{1,\dots,m\} : \frac{|d_j|}{|d_k|} \in (3-\delta, 3)\} \,.$$

Proof of Proposition 6.23. Let $\eta > 0$ be such that

$$1 + \eta = \Big(\frac{3}{3-\delta}\Big)^3 \,.$$

By Lemma 6.26, there exists $\overline{x} = \overline{x}(\eta) = \sum_{k=1}^m \overline{a}_k \mathbf{1}_{A_k}$ with

$$\|x - \overline{x}\| < \eta \|x\| \,, \tag{6.46}$$

such that for all $i, j = 1, \dots, m$, $\frac{|\overline{a}_i|}{|\overline{a}_j|} \notin (1, 1+\eta)$. By symmetry, this means that for all $i, j = 1, \dots, m$,

$$\frac{|\overline{a}_i|}{|\overline{a}_j|} \in \Big(\frac{1}{1+\eta}, 1+\eta\Big) \Longrightarrow |\overline{a}_i| = |\overline{a}_j| \,. \tag{6.47}$$

To prove the proposition we need to construct a simple function $x^\sharp$ such that $\|x - x^\sharp\| < (3/2)\delta$, $\|x\| \le \|x^\sharp\| < (1 + (3/2)\delta)\|x\|$ and

$$S(x^\sharp, k) = \emptyset \text{ for } k = 1, \dots, m \,. \tag{6.48}$$

We will construct $x^\sharp$ satisfying (6.48) inductively. To start the induction, we set for $k = 1, \dots, m$,

$$\overline{a}_k^{(0)} = \overline{a}_k, \quad \overline{x}^{(0)} = \sum_{k=1}^m \overline{a}_k^{(0)} \mathbf{1}_{A_k} = \overline{x}, \quad k_0 = \max(\{k : S(\overline{x}^{(0)}, k) \neq \emptyset\} \cup \{0\}) \,.$$

If $k_0 = 0$, then $\overline{x}^{(0)}$ satisfies (6.48), and we are done. If $k_0 > 0$ then

$$S(\overline{x}^{(0)}, k) = \emptyset \quad \text{for } k > k_0 .$$

Inductively, we will define a sequence of nonnegative integers $k_0 > k_1 > k_2 > \dots$ and a sequence of simple functions $(\overline{x}^{(0)}, \overline{x}^{(1)}, \overline{x}^{(2)}, \dots)$ such that for all n ,

$$S(\overline{x}^{(n)}, k) = \emptyset \text{ for } k > k_n .$$

Once these sequences are defined, we observe that since $k_0 \le m$ and the sequence $(k_n)_n$ is a strictly decreasing sequence of nonnegative integers, there exists $N \le m+1$, such that $k_n = 0$ and $\overline{x}^{(N)}$ satisfies (6.48).

To describe the inductive process, suppose that $(\overline{x}^{(\nu)})_{\nu=0}^n$ and $k_0 > k_1 > \dots > k_n > 0$ have been defined such that for all $\nu \le n$:

$$\overline{x}^{(\nu)} = \textstyle\sum_{k=1}^m \overline{a}_k^{(\nu)} \mathbf{1}_{A_k},$$

$$S(\overline{x}^{(\nu)}, k_\nu) \ne \emptyset \qquad \text{if } k_\nu > 0, \tag{6.49}$$

$$S(\overline{x}^{(\nu)}, k) = \emptyset \qquad \text{for } k > k_\nu, \tag{6.50}$$

$$\frac{\overline{a}_k^{(\nu)}}{\overline{a}_k} \in \left[1, \frac{3}{3-\delta}\right) \qquad \text{for } k = 1, \dots, m, \tag{6.51}$$

$$\overline{a}_k^{(\nu)} = \overline{a}_k \qquad \text{for } k \notin \bigcup_{\alpha=0}^{\nu-1} S(\overline{x}^{(\alpha)}, k_\alpha), \tag{6.52}$$

$$|\overline{a}_k| = |\overline{a}_l| \Longrightarrow |\overline{a}_k^{(\nu)}| = |\overline{a}_l^{(\nu)}|, \tag{6.53}$$

$$|\overline{a}_{k_\nu}| > |\overline{a}_{k_{\nu-1}}| \qquad \text{if } k_\nu > 0. \tag{6.54}$$

Now we define

$$\overline{a}_j^{(n+1)} = \begin{cases} \operatorname{sgn}(\overline{a}_j) \cdot 3 \cdot |\overline{a}_{k_n}^{(n)}| & \text{if } j \in S(\overline{x}^{(n)}, k_n), \\ \overline{a}_j^{(n)} & \text{if } j \notin S(\overline{x}^{(n)}, k_n), \end{cases}$$

$$\overline{x}^{(n+1)} = \sum_{k=1}^m \overline{a}_k^{(n+1)} \mathbf{1}_{A_k},$$

$$k_{n+1} = \max(\{k : S(\overline{x}^{(n+1)}, k) \ne \emptyset\} \cup \{0\}).$$

To prove the induction step we need to show that (6.49)–(6.54) are satisfied for $\nu = n+1$.

Clearly, if $k_{n+1} \ne 0$ then $S(\overline{x}^{(n+1)}, k_{n+1}) \ne \emptyset$ so (6.49) holds for $\nu = n+1$. Similarly, (6.50) holds for $\nu = n+1$ by the definition of k_{n+1}.

To verify (6.51) for $\nu = n+1$ we first observe that if $j \notin S(\overline{x}^{(n)}, k_n)$ then, by definition, $\overline{a}^{(n+1)} = \overline{a}^{(n)}$ and by (6.51) we get

$$\frac{\overline{a}_j^{(n+1)}}{\overline{a}_j} = \frac{\overline{a}_j^{(n)}}{\overline{a}_j} \in \left[1, \frac{3}{3-\delta}\right) \text{ for } j \notin S(\overline{x}^{(n)}, k_n) . \tag{6.55}$$

Thus, it only remains to check that (6.51) is valid for $\nu = n+1$ and $j \in S(\overline{x}^{(n)}, k_n)$. For this we first establish that

$$\overline{a}_j^{(n)} = \overline{a}_j \text{ for } j \in S(\overline{x}^{(n)}, k_n) . \tag{6.56}$$

To prove (6.56), by (6.52), it is enough to show that for all $\alpha,\ 0 \le \alpha < n$,

$$S(\overline{x}^{(n)}, k_n) \cap S(\overline{x}^{(\alpha)}, k_\alpha) = \emptyset .$$

Suppose, for contradiction, that there exist $\alpha,\ 0 \le \alpha < n$ and $i,\ 1 \le i \le m$ such that

$$i \in S(\overline{x}^{(n)}, k_n) \cap S(\overline{x}^{(\alpha)}, k_\alpha) .$$

Now we set $t_1 = t_2 = |\overline{a}_i|,\ \tilde{t}_1 = |\overline{a}_i^{(n)}|,\ \tilde{t}_2 = |\overline{a}_i^{(\alpha)}|,\ t_3 = |\overline{a}_{k_n}|,\ \tilde{t}_3 = |\overline{a}_{k_n}^{(n)}|,\ t_4 = |\overline{a}_{k_\alpha}|,\ \tilde{t}_4 = |\overline{a}_{k_\alpha}^{(\alpha)}|$. Then by (6.51) and Lemma 6.27(i) ,

$$\frac{|\overline{a}_{k_n}|}{|\overline{a}_{k_\alpha}|} = \frac{t_3}{t_4} \in \left(\frac{(3-\delta)^3}{3^3}, \frac{3^3}{(3-\delta)^3} \right) = \left(\frac{1}{1+\eta}, 1+\eta \right) .$$

Thus, by (6.47), $|\overline{a}_{k_n}| = |\overline{a}_{k_\alpha}|$ which contradicts (6.54). Hence, (6.56) is proved.
Next, by (6.56) and by definition of $S(\overline{x}^{(n)}, k_n)$ we see that for $j \in S(\overline{x}^{(n)}, k_n)$

$$\frac{|\overline{a}_j|}{|\overline{a}_{k_n}^{(n)}|} = \frac{|\overline{a}_j^{(n)}|}{|\overline{a}_{k_n}^{(n)}|} \in (3-\delta, 3) .$$

Thus, by Lemma 6.27(ii) with $t_1 = |\overline{a}_j|,\ t_2 = |\overline{a}_{k_n}^{(n)}|,\ t_3 = |\overline{a}_j^{(n+1)}|$, we obtain

$$\frac{\overline{a}_j^{(n+1)}}{\overline{a}_j} = \frac{|\overline{a}_j^{(n+1)}|}{|a_j|} = \frac{t_3}{t_1} \in \left(1, \frac{3}{3-\delta}\right) \quad \text{for} \quad j \in S(\overline{x}^{(n)}, k_n) .$$

Together with (6.55) this ends the proof that (6.51) is satisfied for $\nu = n+1$.
Next we check that (6.52) is valid for $\nu = n+1$, i.e.

$$\overline{a}_k^{(n+1)} = \overline{a}_k \quad \text{for} \quad k \notin \bigcup_{\alpha=0}^{n} S(\overline{x}^{(\alpha)}, k_\alpha) .$$

Let $k \notin \bigcup_{\alpha=0}^{n} S(\overline{x}^{(\alpha)}, k_\alpha)$. Then $k \notin S(\overline{x}^{(n)}, k_n)$ and, by definition, $\overline{a}_k^{(n+1)} = \overline{a}_k^{(n)}$. And since $k \notin \bigcup_{\alpha=0}^{n-1} S(\overline{x}^{(\alpha)}, k_\alpha)$, by (6.52), $\overline{a}_k^{(n)} = \overline{a}_k$. Thus (6.52) holds for $\nu = n+1$.

Our next step is to check (6.53) for $\nu = n+1$. We know, by (6.53), that if $|\overline{a}_k| = |\overline{a}_l|$ then $|\overline{a}_k^{(n)}| = |\overline{a}_l^{(n)}|$. Thus, $k \in S(\overline{x}^{(n)}, k_n)$ if and only if $l \in S(\overline{x}^{(n)}, k_n)$. In either case it follows directly from the definition that $|\overline{a}_k^{(n+1)}| = |\overline{a}_l^{(n+1)}|$, i.e. (6.53) holds for $\nu = n+1$.

Our final step is to verify (6.54) for $\nu = n+1$, i.e. to show that if $k_{n+1} > 0$ then

$$|\overline{a}_{k_{n+1}}| > |\overline{a}_{k_n}| \, .$$

Since $(|\overline{a}_k|)_{k=1}^m$ are arranged in a nonincreasing order and $k_{n+1} = \max(\{k : S(\overline{x}^{(n+1)}, k) \neq \emptyset\} \cup \{0\}) > 0$, it is enough to prove that

$$|\overline{a}_k| \le |\overline{a}_{k_n}| \Longrightarrow S(\overline{x}^{(n+1)}, k) = \emptyset \, . \tag{6.57}$$

If $|\overline{a}_k| = |\overline{a}_{k_n}|$ then by (6.53) for $\nu = n+1$ we get $|\overline{a}_k^{(n+1)}| = |\overline{a}_{k_n}^{(n+1)}|$ and thus,

$$S(\overline{x}^{(n+1)}, k) = S(\overline{x}^{(n+1)}, k_n) \, . \tag{6.58}$$

Notice that $k_n \notin S(\overline{x}^n, k_n)$ so $\overline{a}_{k_n}^{(n+1)} = \overline{a}_{k_n}^{(n)}$. Hence, when $j \notin S(\overline{x}^{(n)}, k_n)$ we get $\overline{a}_j^{(n+1)} = \overline{a}_j^{(n)}$ and

$$\frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_{k_n}^{(n+1)}|} = \frac{|\overline{a}_j^{(n)}|}{|\overline{a}_{k_n}^{(n)}|} \notin (3-\delta, 3) \, .$$

Thus, $j \notin S(\overline{x}^{(n+1)}, k_n)$.

If $j \in S(\overline{x}^{(n)}, k_n)$ then, by definition,

$$\frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_{k_n}^{(n+1)}|} = \frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_{k_n}^{(n)}|} = 3 \notin (3-\delta, 3) \, .$$

Hence, $S(\overline{x}^{(n+1)}, k_n) = \emptyset$ and by (6.58) we see that

$$|\overline{a}_k| = |\overline{a}_{k_n}| \Longrightarrow S(\overline{x}^{(n+1)}, k) = \emptyset \, . \tag{6.59}$$

Now we consider the case

$$|\overline{a}_k| < |\overline{a}_{k_n}| \, .$$

In this case $k > k_n$ and by definition of k_n, $S(\overline{x}^{(n)}, k) = \emptyset$.

By (6.51),

$$\frac{|\overline{a}_k^{(n)}|}{|\overline{a}_k|} \in \left[1, \frac{3}{3-\delta}\right), \qquad \frac{|\overline{a}_{k_n}^{(n)}|}{|\overline{a}_{k_n}|} \in \left[1, \frac{3}{3-\delta}\right) .$$

Thus, if we set $t_1 = |\overline{a}_k|$, $\tilde{t}_1 = |\overline{a}_k^{(n)}|$, $t_2 = |\overline{a}_{k_n}|$, $\tilde{t}_2 = |\overline{a}_{k_n}^{(n)}|$, by Lemma 6.27(iii) we obtain

$$\frac{|\overline{a}_k^{(n)}|}{|\overline{a}_{k_n}^{(n)}|} = \frac{\tilde{t}_1}{\tilde{t}_2} < \frac{9}{(3-\delta)^2} < 3-\delta \, .$$

Thus, $k \notin S(\overline{x}^{(n)}, k_n)$ and, by definition, $\overline{a}_k^{(n+1)} = \overline{a}_k^{(n)}$. Further, for all $j \notin S(\overline{x}^{(n)}, k_n)$ we have $\overline{a}_j^{(n+1)} = \overline{a}_j^{(n)}$, and therefore

$$\frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_k^{(n+1)}|} = \frac{|\overline{a}_j^{(n)}|}{|\overline{a}_k^{(n)}|} \notin (3-\delta, 3) ,$$

since $S(\overline{x}^{(n)}, k) = \emptyset$. Hence,

$$S(\overline{x}^{(n+1)}, k) \subset S(\overline{x}^{(n)}, k_n) .$$

But if $j \in S(\overline{x}^{(n+1)}, k) \cap S(\overline{x}^{(n)}, k_n)$ then

$$\frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_{k_n}^{(n)}|} = 3, \quad \frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_k^{(n)}|} = \frac{|\overline{a}_j^{(n+1)}|}{|\overline{a}_k^{(n+1)}|} \in (3-\delta, 3) ,$$

and if we set $t_1 = |\overline{a}_j^{(n+1)}|$, $t_2 = |\overline{a}_{k_n}|$, $\tilde{t}_2 = |\overline{a}_{k_n}^{(n)}|$, $t_3 = |\overline{a}_k|$, $\tilde{t}_3 = |\overline{a}_k^{(n)}|$ then by (6.51) and by Lemma 6.27(iv), we get

$$\frac{|\overline{a}_{k_n}|}{|\overline{a}_k|} = \frac{t_2}{t_3} \in \Big(\frac{(3-\delta)^2}{9}, \frac{3}{3-\delta}\Big) \subset \Big(\frac{1}{1+\eta}, 1+\eta\Big) .$$

Hence, by (6.47), $|\overline{a}_{k_n}| = |\overline{a}_k|$ which contradicts our assumption that $|\overline{a}_{k_n}| > |\overline{a}_k|$.

Thus, $S(\overline{x}^{(n+1)}, k) = \emptyset$ if $|\overline{a}_k| < |\overline{a}_{k_n}|$, which together with (6.59) concludes the proof of (6.57) and (6.54) for $\nu = n+1$.

Note that (6.54) for $\nu = n+1$, implies that $k_{n+1} < k_n$.

This ends the proof of the inductive process.

To finish the proof of the proposition we notice, as indicated above, that since $(k_n)_{n\geq 0}$ is a strictly decreasing sequence of nonnegative integers, it must be finite, i.e. there exists $N \leq m+1$, such that $k_N = 0$.

Set $x^\sharp = \overline{x}^{(N)}$. By (6.50), $S(\overline{x}^{(N)}, k) = \emptyset$, for all $k > k_N = 0$, and, by (6.51),

$$\begin{aligned} \|x^\sharp - \overline{x}\| &= \Big\| \sum_{k=1}^m (\overline{a}_k^{(N)} - \overline{a}_k)\mathbf{1}_{A_k} \Big\| = \Big\| \sum_{k=1}^m \overline{a}_k \Big(\frac{\overline{a}_k^{(N)}}{\overline{a}_k} - 1\Big)\mathbf{1}_{A_k} \Big\| \\ &\leq \Big(\frac{3}{3-\delta} - 1\Big) \cdot \Big\| \sum_{k=1}^m \overline{a}_k \mathbf{1}_{A_k} \Big\| = \frac{\delta}{3-\delta}\|\overline{x}\|. \end{aligned}$$

Thus, by (6.46), when $\delta < 1/8$ we have

$$\begin{aligned} \|x^\sharp - x\| &\leq \|x^\sharp - \overline{x}\| + \|\overline{x} - x\| \leq \frac{\delta}{3-\delta}\|\overline{x}\| + \eta\|x\| \leq \frac{\delta}{3-\delta}(1+\eta)\|x\| + \eta\|x\| \\ &= \Big[\frac{\delta}{3-\delta} \cdot \Big(\frac{3}{3-\delta}\Big)^3 + \Big(\frac{3}{3-\delta}\Big)^3 - 1\Big]\|x\| = \frac{81 - (3-\delta)^4}{(3-\delta)^4}\,\|x\| \\ &< \frac{3}{2}\,\delta\|x\|. \end{aligned}$$

Finally note that, by (6.51), for all $j = 1, \dots, m$, $\frac{\overline{a}_j^{(N)}}{\overline{a}_j} \geq 1$. Combining the last two inequalities and Lemma 6.26 we get

$$\Big(1 + \frac{3}{2}\delta\Big)\|x\| > \|x^{\sharp}\| = \|\overline{x}^{(N)}\| \geq \|\overline{x}\| \geq \|x\| . \qquad \square$$

6.4 Near isometric classification of $L_p(\mu)$-spaces for $1 \leq p < \infty$, $p \neq 2$

Recall that the *Banach–Mazur distance* $d(X, Y)$ between Banach spaces X and Y is given by

$$d(X, Y) = \inf\big\{\|T\|\|T^{-1}\| : T \in \mathcal{L}(X, Y) \text{ is an onto isomorphism}\big\} .$$

If X and Y are not isomorphic, we say that $d(X, Y) = \infty$.

The main result of this section is the following theorem.

Theorem 6.28. *Let* $1 \leq p < \infty$, $p \neq 2$, *and* $k_p > 1$, $k_1 = 2$ *be the constants defined in Corollary* 6.12 *(*$k_p = 1 + \delta_p(1)$*). Let* $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$ *be finite atomless measure spaces with the Maharam sets* $\mathcal{M}_i$. *Suppose that*

$$d\big(L_p(\mu_1), L_p(\mu_2)\big) < k_p .$$

Then $\mathcal{M}_1 = \mathcal{M}_2$.

A similar result for separable (not necessarily atomless) $L_p(\mu)$-spaces was obtained by Benyamini in [13], and the same result for $p = 1$ and not necessarily separable spaces was proved by Cambern in [24].

Let $\mathcal{M}$ be the Maharam set of a finite atomless measure space (Ω, Σ, μ). By the Maharam theorem, there exists a decomposition $\Omega = \bigsqcup_{\alpha\in\mathcal{M}} \Omega_\alpha$ and a collection of positive numbers $(\varepsilon_\alpha)_{\alpha\in\mathcal{M}}$ such that the measure spaces $(\Omega_\alpha, \Sigma(\Omega_\alpha), \mu|_{\Sigma(\Omega_\alpha)})$ and $\varepsilon_\alpha \cdot D^{\omega_\alpha}$ are isomorphic for every $\alpha \in \mathcal{M}$. In particular,

$$L_p(\mu) = \Big(\sum_{\alpha\in\mathcal{M}} L_p(\Omega_\alpha)\Big)_p ,$$

where $L_p(\Omega_\alpha)$ and $L_p(D^{\omega_\alpha})$ are isometrically isomorphic for every $p \in (0, +\infty)$.

Let P be a projection on $L_p(\mu)$. Denote by $\mathcal{M}(P)$ the set of all those $\alpha \in \mathcal{M}$ for which the image im P of P is rich with respect to $L_p(\Omega_\alpha)$, and set

$$X(P) = \Big(\sum_{\alpha\in\mathcal{M}(P)} L_p(\Omega_\alpha)\Big)_p, \quad Y(P) = \Big(\sum_{\alpha\in\mathcal{M}\setminus\mathcal{M}(P)} L_p(\Omega_\alpha)\Big)_p .$$

For subspaces X and Y of a Banach space Z we consider the following asymmetric function

$$\rho(X,Y) = \inf\{\|x - y\| : x \in S_X, y \in Y\} . \tag{6.60}$$

Our main tool will be the following lemma.

Lemma 6.29. *If $\|P\| < k_p$, then either $X(P) = \{0\}$, or $\rho(\ker P, X(P)) > 0$.*

Proof of Lemma 6.29. Suppose that $X(P) \ne \{0\}$ and $\rho(\ker P, X(P)) = 0$. We prove that $\|P\| \ge k_p$. First we prove that the restriction $(I - P)|_{X(P)}$ is a narrow operator.

Fix any measurable subset $A \subseteq \bigcup_{\alpha\in\mathcal{M}(P)} \Omega_\alpha$ and $\varepsilon > 0$. Since $\operatorname{im} P$ is rich with respect to $\bigcup_{\alpha\in\mathcal{M}(P)} \Omega_\alpha$, we can find a mean zero sign x on A and $y \in \operatorname{im} P$ such that

$$\|x - y\| < \frac{\varepsilon}{\|P\| + 1} .$$

Since $Py = y$, we obtain

$$\big\|(I-P)x\big\| = \|x-Px\| \le \|x-Py\|+\|Py-Px\| < \frac{\varepsilon}{\|P\| + 1}+\|P\|\|x-Py\| < \varepsilon.$$

Thus, $(I - P)|_{X(P)}$ is narrow.

Now we prove that $\|(I - P)|_{X(P)}\| \ge 1$. Fix $\varepsilon > 0$. Since $\rho(\ker P, X(P)) = 0$, there exist $y \in \ker P$, $\|y\| = 1$, and $x \in X(P)$ such that $\|x - y\| < \varepsilon$. Thus

$$\begin{aligned}\big\|(I - P)x\big\| &\ge \big\|(I - P)\, y\big\| - \big\|(I - P)\, x - (I - P)y\big\| \\ &= \|y\| - \big\|x - y + P(y - x)\big\| \ge 1 - \|x - y\|\big(\|P\| + 1\big) \\ &> 1 - \varepsilon\big(\|P\| + 1\big)\end{aligned} \tag{6.61}$$

and

$$\|x\| \le \|y\| + \|x - y\| < 1 + \varepsilon . \tag{6.62}$$

Dividing (6.61) by (6.62), we obtain

$$\big\|(I - P)\big|_{X(P)}\big\| \ge \frac{\big\|(I - P)x\big\|}{\|x\|} \ge \frac{1 - \varepsilon\big(\|P\| + 1\big)}{\|y\| + \|x - y\|} \ge \frac{1 - \varepsilon\big(\|P\| + 1\big)}{1 + \varepsilon} .$$

By arbitrariness of $\varepsilon > 0$, we deduce that $\|(I - P)|_{X(P)}\| \ge 1$.

By Corollary 6.4 for $p = 1$, and by Theorem 6.8 for $p \in (1, 2) \cup (2, +\infty)$, we get

$$\begin{aligned}\|P\| &\ge \big\|P\big|_{X(P)}\big\| = \big\|I\big|_{X(P)} - (I - P)\big|_{X(P)}\big\| \\ &\ge 1 + \delta_p\Big(\big\|(I - P)\big|_{X(P)}\big\|\Big) \ge 1 + \delta_p(1) = k_p .\end{aligned}$$

□

Corollary 6.30. *If $\|P\| < k_p$, then* $\ker P$ *embeds in $Y(P)$.*

Proof. By Lemma 6.29, either $X(P) = \{0\}$, or $\rho(\ker P, X(P)) > 0$. If $X(P) = \{0\}$ then the assertion of the corollary is evident. Suppose that $\rho(\ker P, X(P)) > 0$. We denote by Q the projection from $L_p(\mu)$ onto $Y(P)$ with $\ker Q = X(P)$, and show that the restriction $Q|_{\ker P}$ is an isomorphic embedding of $\ker P$ in $Y(P)$.

Indeed, let $x \in \ker P$ be any element with $\|x\| = 1$. Then

$$\|Qx\| = \|x - (I - Q)x\| \geq \rho\big(\ker P, X(P)\big) ,$$

by the definitions of ρ and Q. Thus, the operator $Q|_{\ker P}$ is bounded from below. □

Corollary 6.31. *Suppose that* $\|P\| < k_p$, *and let*

$$\mathcal{M}' = \big\{\alpha \in \mathcal{M} : \aleph_\alpha > \operatorname{dens}\ker P\big\}, \quad \mathcal{M}'' = \mathcal{M} \setminus \mathcal{M}' .$$

Then either $\mathcal{M}' = \emptyset$, *or*

$$\rho\Big(\ker P, \Big(\sum_{\alpha\in\mathcal{M}'} L_p(\Omega_\alpha)\Big)_p\Big) > 0 .$$

Moreover, $\ker P$ *embeds in* $(\sum_{\alpha\in\mathcal{M}''} L_p(\Omega_\alpha))_p$.

Proof. By Lemma 6.29 and Corollary 6.30 it is enough to prove that $\mathcal{M}' \subseteq \mathcal{M}(P)$. Let $\alpha \in \mathcal{M}'$, then, $\aleph_\alpha > \operatorname{dens}\ker P$. By Theorem 2.12, the restriction operator $(I - P)|_{L_p(\Omega_\alpha)}$ is narrow. This exactly means that $\operatorname{im} P$ is rich with respect to $L_p(\Omega_\alpha)$, that is, $\alpha \in \mathcal{M}(P)$. □

Lemma 6.32. *Let* $P \neq I$ *be a projection on* $L_p(\mu)$ *with* $\|P\| < k_p$. *Then*

$$\operatorname{dens}\ker P \in \big\{\aleph_\alpha : \alpha \in \overline{\mathcal{M}}\big\} ,$$

where $\overline{\mathcal{M}}$ *is the closure of* $\mathcal{M}$ *in the order topology. Conversely, if* $\beta \in \overline{\mathcal{M}}$, *then there exists a projection* P *of* $L_p(\mu)$ *of norm one such that* $\operatorname{dens}\ker P = \aleph_\beta$.

Proof of Lemma 6.32. Let $\mathcal{M}'' = \{\alpha \in \mathcal{M} : \operatorname{dens}\ker P \geq \aleph_\alpha\}$. Since $\mathcal{M}'' \subseteq \mathcal{M}$ and $\mathcal{M}$ is, at most, countable

$$\sum_{\alpha\in\mathcal{M}''} \aleph_\alpha \leq \operatorname{dens}\ker P .$$

On the other hand, since $\ker P$ embeds in $(\sum_{\alpha\in\mathcal{M}''} L_p(\Omega_\alpha))_p$ whose dimension equals $\sum_{\alpha\in\mathcal{M}''} \aleph_\alpha$, by Corollary 6.31, we get

$$\sum_{\alpha\in\mathcal{M}''} \aleph_\alpha \geq \operatorname{dens}\ker P$$

Thus,

$$\operatorname{dens}\ker P = \sum_{\alpha\in\mathcal{M}''} \aleph_\alpha \in \big\{\aleph_\alpha : \alpha \in \overline{\mathcal{M}}\big\} .$$

Suppose that $\beta \in \overline{\mathcal{M}}$, say, $\beta = \lim_n \alpha_n$, where $\alpha_n \in \mathcal{M}$. Then

$$\operatorname{dens}\Big(\sum_n L_p(\Omega_{\alpha_n})\Big)_p = \sum_n \aleph_\alpha = \aleph_\beta \ ,$$

and the subspace $(\sum_n L_p(\Omega_{\alpha_n}))_p$ is the kernel of the projection defined by $Px = x \cdot \mathbf{1}_B$ for each $x \in L_p(\mu)$, where $B = \Omega \setminus \bigcup_n \Omega_{\alpha_n}$. □

A subspace $X \subseteq L_p(\mu)$ is isometrically isomorphic to an $L_p(\nu)$-space if and only if X is the range of norm one from $L_p(\mu)$ onto X (Ando [9], and Douglas [33]). Thus Lemma 6.32 has the following corollary.

Corollary 6.33. *Let X be a subspace of $L_p(\mu)$ which is isometrically isomorphic to some $L_p(\nu)$-space where ν is not necessarily an atomless measure. Then*

$$\operatorname{codim} X \stackrel{\text{def}}{=} \operatorname{dens} L_p(\mu)/X \in \left\{\aleph_\alpha : \alpha \in \overline{\mathcal{M}}\right\} .$$

Conversely, if $\beta \in \overline{\mathcal{M}}$ then there exists a subspace X of $L_p(\mu)$ which is isometrically isomorphic to some $L_p(\nu)$-space such that $\operatorname{codim} X = \aleph_\beta$.

Lemma 6.34. *Assume $1 \le p < \infty$, $p \ne 2$ and let α be an ordinal. Suppose that $L_p(D^{\omega_\alpha})$ embeds in $X = (\sum_{j\in J} X_j)_p$, where J is some infinite set of indices and the X_j are Banach spaces. Then $\aleph_\alpha \le \operatorname{dens} X_j$, for some $j \in J$.*

Proof of Lemma 6.34. Note that $L_p(D^{\omega_\alpha})$ contains a Rademacher system $(r_\gamma)_{\gamma<\omega_\alpha}$ of cardinality $\aleph_\alpha$. Indeed, given $\gamma < \omega_\alpha$, we define $r_\gamma(t) = \theta_\gamma$ for any $t = (\theta_\beta)_{\beta<\omega_\alpha} \in D^{\omega_\alpha}$. By the Khintchine inequality [79, p. 66], $\ell_2(\omega_\alpha)$ embeds in $L_p(D^{\omega_\alpha})$, and hence, in X. Let Y be a subspace of X isomorphic to $\ell_2(\omega_\alpha)$.

Assume, on the contrary, that $\operatorname{dens} X_j < \aleph_\alpha$ for all $j \in J$. We are going to construct a subspace of Y isomorphic to ℓ_p, which will give us the contradiction since $p \ne 2$.

We construct inductively sequences $x_n \in X$ and $y_n \in Y$, $n \in \mathbb{N}$, and a sequence (J_n) of finite disjoint subsets $J_n \subset J$ as follows. Choose any $y_1 \in Y$ with $\|y_1\| = 1$. Then find a finite subset $J_1 \subset J$ and $x_1 \in \operatorname{lin}(X_j : j \in J_1)$ such that $\|x_1 - y_1\| < 2^{-1}$. Suppose that finite pairwise disjoint subsets $J_k \subset J$ and elements $x_k \in X$, $y_k \in Y$, $k = 1, \dots, n$, have been chosen so that $\|x_k\| = 1$, $\|x_k - y_k\| < 2^{-k}$ and $x_k \in \operatorname{lin}(x_j : j \in J_k)$. Since I_n is finite, by the assumption that $\operatorname{dens} X_j < \aleph_\alpha$ for every $j \in J$, we have that

$$\operatorname{dens} \operatorname{lin}\Big(X_j : j \in \bigcup_{k=1}^{n} J_k\Big) < \aleph_\alpha = \operatorname{dens} Y \ . \tag{6.63}$$

We claim that there exists a finite subset $J_{n+1} \subset J \setminus \bigcup_{k=1}^n J_k$, $x_{n+1} \in \operatorname{lin}(X_j : j \in \bigcup_{k=1}^n J_k)$ and $y_{n+1} \in Y$ so that $\|x_{n+1}\| = 1$ and $\|x_{n+1} - y_{n+1}\| < 2^{-n-1}$.

Indeed, if this were not true, then the quotient map of X by the subspace $(\sum_{j\in J\setminus\bigcup_{k=1}^{n} J_k} X_j)_p$ would be bounded from below, which contradicts (6.63). Thus, the inductive construction has been done.

By the construction, the sequence (x_n) is isometrically equivalent to the unit vector basis of ℓ_p, and by the Krein–Milman–Rutman theorem on the stability of basic sequences ([52, p. 64], [79, p. 5]), the inequality $\|x_n - y_n\| < 2^{-n}$ for each $n \in \mathbb{N}$ implies that the sequence (y_n) is equivalent to the unit vector basis of ℓ_p. This contradicts the assumptions that Y is isomorphic to $\ell_2(\omega_\alpha)$, and $p \neq 2$. □

Proof of Theorem 6.28. Using Maharam's theorem, for $i = 1, 2$, we decompose $\Omega_i = \bigsqcup_{\alpha\in\mathcal{M}_i} \Omega_{i,\alpha}$ so that $L_p(\Omega_{i,\alpha})$ and $L_p(D^{\omega_\alpha})$ are isometrically isomorphic for every $\alpha \in \mathcal{M}_i$.

Fix any $\alpha_1 \in \mathcal{M}_1$. Observe that the operator defined by $Px = x - x\cdot\mathbf{1}_{\Omega_{1,\alpha_1}}$ for each $x \in L_p(\mu_1)$, is a projection of norm one from $L_p(\mu_1)$ onto the subspace

$$\Big(\sum_{\alpha\in\mathcal{M}_1\setminus\{\alpha_1\}} L_p\big(\Omega_{1,\alpha}\big)\Big)_p$$

and with $\ker P = L_p(\Omega_{1,\alpha_1})$. Since $d(L_p(\mu_1), L_p(\mu_2)) < k_p$, there exists a projection Q of $L_p(\mu_2)$ of norm $\|Q\| < k_p$ with $\ker Q$ isomorphic to $L_p(D^{\omega_{\alpha_1}})$. By Corollary 6.31, $\ker Q$ embeds in

$$\Big(\sum_{\alpha\in\mathcal{M}_2''} L_p\big(\Omega_{2,\alpha}\big)\Big)_p,$$

where $\mathcal{M}_2'' = \{\alpha \in \mathcal{M}_2 : \alpha \le \alpha_1\}$. By Lemma 6.34, there exists $\alpha_2 \in \mathcal{M}_2''$ such that $\alpha_1 \le \alpha_2$. Hence, $\alpha_1 = \alpha_2$ by the definition of $\mathcal{M}_2''$. Thus, $\alpha_1 \in \mathcal{M}_2$. By arbitrariness of $\alpha_1 \in \mathcal{M}_1$, we obtain that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The inverse inclusion can be proved analogously. □

Chapter 7

Strict singularity versus narrowness

We saw that every compact operator is narrow, and in Chapter 2 we identified other classes of "small" operators which are narrow. The next natural ideal of "small" operators to consider is the ideal of strictly singular operators.

Recall that an operator $T : X \to Y$ is called strictly singular if for every infinite dimensional subspace X_1 of X, the restriction of T to X_1 is not an isomorphic embedding, that is, for every $\varepsilon > 0$ and every infinite dimensional subspace X_1 of X, there exists an element $x \in X_1$ with $\|Tx\| < \varepsilon\|x\|$. Superficially, this condition appears similar to the definition of a narrow operator. However there are important differences on the conditions that an element x with $\|Tx\| < \varepsilon\|x\|$ has to satisfy: a symmetric sign on an arbitrary set, versus an element of an arbitrary infinite dimensional subspace. These differences are fundamental and, as we saw in Chapter 4, there do exist nonstrictly singular operators, and even Enflo operators, which are narrow. In this chapter we investigate the converse problems whether some kind of strict singularity implies narrowness. These problems were posed by Plichko and Popov in [110], and they were already mentioned in Chapter 2, Open problems 2.6 and 2.7. We list them here again in a somewhat expanded version.

Open problem 7.1. Let T be an operator from E to X. Does T have to be narrow, provided that

(a) T is strictly singular?

(b) T is $Z-$strictly singular for an appropriately chosen infinite dimensional subspace Z of E?

(c) T is non-Enflo, that is, T is E-strictly singular?

A number of other notions closely related to strict singularity are considered in the literature. These include strictly cosingular operators, introduced in [106], superstrictly singular operators, also known as finitely strictly singular operators, introduced implicitly in [98] and explicitly in [96] and [97], disjointly strictly singular operators, introduced in [47], and Schreier strictly singular operators, introduced in [8]. All these notions are actively studied and it would be very interesting to discover their relationships with narrowness.

Problems 7.1(a)–(c) are open in general, but, somewhat surprisingly, they do have affirmative answers in important special cases.

Bourgain and Rosenthal [20] showed that every ℓ_1-strictly singular operator from L_1 to any Banach space X is narrow (Theorem 7.2 below). Moreover, they described sufficient conditions for an operator $T \in \mathcal{L}(L_1, X)$ to fix a copy of ℓ_1 (Theorem 7.4 below). Section 7.1 is devoted to the proofs of these results.

Open problem 7.1(b) is very interesting for the case when $Z = \ell_2$. It was posed in that form by Plichko and Popov in [110], cf. Open problem 2.7. It has two positive partial answers. Flores and Ruiz [39] proved that every ℓ_2-strictly singular regular operator (i.e. a difference of two positive operators) from $L_p[0,1]$, $1 \le p < \infty$, to an order continuous Banach lattice F, is narrow. We present a generalized version of this result in Section 10.9. The authors jointly with Mykhaylyuk and Schechtman [102] proved that every ℓ_2-strictly singular operator from $L_p[0,1]$, $1 \le p < \infty$, to a Banach space X with an unconditional basis, is narrow. We present this result in Section 9.5.

Problem 7.1(c) has a very strong affirmative answer for operators on L_1, which follows from Rosenthal's [128] very deep and remarkable characterization of narrow operators on L_1 (Theorem 7.30 below). This fundamental result connects several deep notions in the theory. We present it in its general context in Section 7.2. We show the connections with pseudo-embeddings on L_1 (see Definition 1.32), including Rosenthal's characterization that an operator on L_1 is a pseudo-embedding if and only if it acts as an almost isometry on a subspace of the form $L_1(A)$ for a suitable subset $A \subseteq [0,1]$ ([128], Theorem 7.39 below). We then present ideas related to the Enflo–Starbird maximal function λ, and, as a culmination, the theorem asserting that the notions of narrow, pseudonarrow and λ-narrow operators all coincide for operators on L_1, and in addition they are equivalent to the fact that for each measurable set A, the restriction of the operator to $L_1(A)$ is not an isomorphic embedding (Theorem 7.45). This combines results of Enflo and Starbird [37], Kalton [66] and Rosenthal [128]. As a first corollary, we present the fact that narrow operators on L_1 form a band ([92, 93], see Theorem 7.46). In particular, this yields that a sum of two narrow operators on L_1 is narrow, a fact having a long history: first it was asserted in [110] with a wrong proof, and then proved in [132] and [58] in different ways.

Problem 7.1(c) has a negative answer for operators on L_p, for $p > 2$ (see Example 7.56), but it has an affirmative answer for operators on L_p, for $1 < p < 2$, as proved by Johnson, Maurey, Schechtman and Tzafriri [49] (Theorem 7.55 below). We present the proof of this result in Section 7.3. We do not know whether Theorem 7.30 can be extended to L_p, for $1 < p < 2$ (see Open problem 7.52). A partial answer to this question is presented in Section 11.2.

The last three sections of this chapter are devoted to geometric applications of the above results. In Section 7.4, as an application of Theorem 7.46, we prove that if a subspace of L_1 is isomorphic to L_1 then it contains an almost isometric copy of L_1. We also obtain a result about operators from L_1 to any Banach space X which fix a copy of L_1. This will have an important application in Section 8.5. In Section 7.5 we show that "well" co-complemented subspaces of L_p are isomorphic to L_p (see Corollary 7.83). In Section 7.6 we show that any rich subspace X of L_1 satisfies Daugavet property for L_1-strictly singular operators and for operators narrow on X (see Definition 7.84).

7.1 Bourgain–Rosenthal's theorem on narrowness of ℓ_1-strictly singular operators

This section is devoted to the following elegant result of Bourgain and Rosenthal.

Theorem 7.2 ([20]). *Let X be a Banach space. Then every ℓ_1-strictly singular operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Theorem 7.2 immediately generalizes to arbitrary atomless $L_1(\mu)$-spaces.

Corollary 7.3. *Let (Ω, Σ, μ) be a finite atomless measure space, and let X be a Banach space. Then every ℓ_1-strictly singular operator $T \in \mathcal{L}(L_1(\mu), X)$ is narrow.*

Theorem 7.2 is obtained as a corollary of the following more general result.

Theorem 7.4 (Bourgain, Rosenthal [20]). *Let X be a Banach space. Assume that an operator $T \in \mathcal{L}(L_1, X)$ satisfies the following property:*

There exists $\delta > 0$ such that $\|Tx\| \geq \delta$, for every sign $x = \mathbf{1}_A - \mathbf{1}_B$, where A and B are finite unions of dyadic intervals with $A \sqcup B = [0, 1]$.

Then T fixes a copy of ℓ_1.

It is interesting to remark that the assumption on T in Theorem 7.4 does not imply that T is not narrow. Indeed, Rosenthal in [125] constructed a Banach space X with an unconditional basis, and an operator T satisfying the assumption from Theorem 7.4. On the other hand, by Theorem 11.11 (see below), T is narrow. This confirms that Theorem 7.4 is essentially stronger than Theorem 7.2.

For the proof of Theorem 7.4 we need a chain of lemmas with preliminary definitions and notation.

Special terminology and notation

We will use trees of sets in the sense of Definition 1.3, however, with another notation.

We denote by $\mathcal{D}$ the set of all finite sequences of 0s and 1s. Given any $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{D}$, we set $|\alpha| = k$. Empty sequence is also considered to be an element of $\mathcal{D}$ with $|\emptyset| = 0$. For any $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{D}$ and $\beta = (\beta_1, \dots, \beta_j) \in \mathcal{D}$ we say that $\alpha \leq \beta$ if and only if $k \leq j$ and $\alpha_i = \beta_i$ for every $i = 1, \dots, k$. We denote $\alpha\, 0 = (\alpha_1, \dots, \alpha_k, 0)$, $\alpha\, 1 = (\alpha_1, \dots, \alpha_k, 1)$ and $\alpha\, \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_j)$.

Definition 7.5. A *tree of sets* is a collection $(G_\alpha)_{\alpha \in \mathcal{D}}$ of sets $G_\alpha \in \Sigma$ such that

$$G_\alpha = G_{\alpha\, 0} \sqcup G_{\alpha\, 1} \quad \text{and} \quad \mu(G_\alpha) = 2^{-|\alpha|} \mu(G_\emptyset)\,.$$

Definition 7.6. A *tree of signs* is a collection $(g_\alpha)_{\alpha \in \mathcal{D}}$ of signs $g_\alpha \in L_1$ such that $(\operatorname{supp} g_\alpha)_{\alpha \in \mathcal{D}}$ is a tree of sets.

By the support of a tree of signs $\mathcal{G} = (g_\alpha)_{\alpha \in \mathcal{D}}$ we will mean the set $\operatorname{supp} \mathcal{G} = G_\emptyset$. For every $x \in L_1 \setminus \{0\}$, by $\widetilde{x}$ we denote $x/\|x\|$.

Definition 7.7. Let $\mathcal{G}$ be a tree of signs. We say that a sign $x \in L_1$ is a *$\mathcal{G}$-function* if $x = g_\alpha$ or $x = -g_\alpha$ for some $\alpha \in \mathcal{D}$. Given $\varepsilon > 0$, a sign $x \in L_1$ is called an *ε-$\mathcal{G}$-function* if there exists a $\mathcal{G}$-function y with $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$.

Definition 7.8. A finite sum of disjoint $\mathcal{G}$-functions is called an *elementary $\mathcal{G}$-sign*. Given $\varepsilon > 0$, a sign $x \in L_1$ is called an *ε-elementary $\mathcal{G}$-sign* if there exists an elementary $\mathcal{G}$-sign y with $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$.

Definition 7.9. Let $\mathcal{G}' = (g'_\alpha)_{\alpha \in \mathcal{D}}$ and $\mathcal{G}'' = (g''_\alpha)_{\alpha \in \mathcal{D}}$ be trees of signs. $\mathcal{G}'$ is said to be *related to* $\mathcal{G}''$ if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for each $\alpha \in \mathcal{D}$ with $|\alpha| \geq k$, g'_α is an ε-elementary $\mathcal{G}''$-sign. $\mathcal{G}'$ is called a *piece* of $\mathcal{G}''$ if there exists $\beta \in \mathcal{D}$ such that $g'_\alpha = g''_{\beta\,\alpha}$ for every $\alpha \in \mathcal{D}$.

Given a nonempty subset $M \subseteq B_{L_\infty}$ and $x \in L_1$, we set

$$M(x) = \sup_{y \in M} \left| \int_{[0,1]} x y \, \mathrm{d}\mu \right| .$$

Clearly, M is a seminorm on L_1 such that $M(x) \leq \|x\|$ for every $x \in L_1$.

Definition 7.10. Let $b > 0$. We say that a nonempty subset $M \subseteq B_{L_\infty}$ *b-norms* an element $x \in L_1$ if $M(x) \geq b$. We say that M *b-norms* a tree of signs $\mathcal{G}$ if M b-norms every elementary $\mathcal{G}$-sign. We say that M *b^+-norms* $\mathcal{G}$ if there exists $\varepsilon > 0$ such that M $(b + \varepsilon)$-norms $\mathcal{G}$.

Observe that if M b-norms $\mathcal{G}$, then M $(b - \varepsilon)$-norms every ε-elementary $\mathcal{G}$-sign for every $\varepsilon \in (0, b)$.

Let $0 < a < b$, $M \subseteq B_{L_\infty}$ and $x \in L_1$, $x \neq 0$. We set

$$M^{x>b} = \left\{ m \in M : \left| \int_{[0,1]} m\widetilde{x} \, \mathrm{d}\mu \right| > b \right\} ,$$

$$M^{x<a} = \left\{ m \in M : \left| \int_{[0,1]} m\widetilde{x} \, \mathrm{d}\mu \right| < a \right\} .$$

Proof of Theorem 7.4

By an ℓ_1-sequence we mean a sequence equivalent to the unit vector basis of ℓ_1.

Lemma 7.11. *Let $\mathcal{G}$ be a tree of signs. Then for every $n \in \mathcal{N}$ there exists a system $(h_i)_{i=1}^n$ of elementary $\mathcal{G}$-signs with $\operatorname{supp} h_i = \operatorname{supp} \mathcal{G}$, for $i = 1, \ldots, n$ and $\int_{[0,1]} h_i h_j \, \mathrm{d}\mu = 0$, for $i \neq j$.*

Proof. Fix any $n \in \mathbb{N}$ and set for each $i = 1, \dots, n$ and $k = 1, \dots, 2^n$

$$\theta_{i,k} = r_i\Big(\frac{k}{2^n}\Big) = (-1)^{\left[\frac{k-1}{2^{n-i}}\right]},$$

where (r_i) is the standard Rademacher system and $[a]$ indicates the integer part of $a \in \mathbb{R}$. By the well-known property of the Rademacher system, if $i \neq j$ then

$$2^{-n} \sum_{k=1}^{2^n} \theta_{i,k}\theta_{j,k} = \int_{[0,1]} r_i r_j \,\mathrm{d}\mu = 0\,. \tag{7.1}$$

Let $\mathcal{G} = (g_\alpha)_{\alpha \in \mathcal{D}}$, and $g_1, \dots, g_{2^n}$ be an enumeration of $\{g_\alpha : |\alpha| = n\}$. Observe that $(g_k)_{k=1}^{2^n}$ is a disjoint system with $\bigcup_{k=1}^{2^n} \operatorname{supp} g_k = \operatorname{supp} \mathcal{G}$. Define, for $i = 1, \dots, n$,

$$h_i = \sum_{k=1}^{2^n} \theta_{i,k} g_k\,.$$

By disjointness of g_ks and (7.1), for $i \neq j$ we have

$$\int_{[0,1]} h_i h_j \,\mathrm{d}\mu = \sum_{k=1}^{2^n} \int_{[0,1]} \theta_{i,k}\theta_{j,k}\, g_k^2 \,\mathrm{d}\mu = \frac{\mu(\operatorname{supp}\mathcal{G})}{2^n} \sum_{k=1}^{2^n} \theta_{i,k}\theta_{j,k} = 0\,.$$

□

Lemma 7.12. *Let x, y be signs and $C = \{t \in [0,1] : x(t) = y(t) \neq 0\}$. Then for $A = \operatorname{supp} x$ and $B = \operatorname{supp} y$ we have*

$$\|\widetilde{x} - \widetilde{y}\| = 2\Big(1 - \frac{\mu(C)}{\max\{\mu(A), \mu(B)\}}\Big)\,.$$

Consequently, for every $\lambda > 0$ we have

$$\|\widetilde{x} - \widetilde{y}\| < \lambda \quad \textit{if and only if} \quad \mu(C) > \Big(1 - \frac{\lambda}{2}\Big) \max\{\mu(A), \mu(B)\}\,.$$

Proof. Without loss of generality we may and do assume that $\mu(B) \leq \mu(A)$. Let $W = (A \cap B) \setminus C$, that is, $W = \{t \in [0,1] : x(t) = -y(t) \neq 0\}$. Then

$$\begin{aligned}
\|\widetilde{x} - \widetilde{y}\| &= \int_{A \setminus B} |\widetilde{x} - \widetilde{y}| \,\mathrm{d}\mu + \int_{B \setminus A} |\widetilde{x} - \widetilde{y}| \,\mathrm{d}\mu + \int_C |\widetilde{x} - \widetilde{y}| \,\mathrm{d}\mu + \int_W |\widetilde{x} - \widetilde{y}| \,\mathrm{d}\mu \\
&= \frac{\mu(A \setminus B)}{\mu(A)} + \frac{\mu(B \setminus A)}{\mu(B)} + \mu(C)\Big(\frac{1}{\mu(B)} - \frac{1}{\mu(A)}\Big) + \mu(W)\Big(\frac{1}{\mu(B)} + \frac{1}{\mu(A)}\Big) \\
&= 2 - 2\,\frac{\mu(C)}{\mu(A)}\,.
\end{aligned}$$

□

Lemma 7.13. *Let x and y be nonzero signs and $\varepsilon \in (0,1)$.*

(a) *If $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$ then $\|x - y\| < 2\varepsilon \min\{\|x\|, \|y\|\}$.*

(b) *If $\|x - y\| < \varepsilon\|x\|$ then $\|\widetilde{x} - \widetilde{y}\| < 2\varepsilon$.*

Proof. Let A, B, C, W be as defined in Lemma 7.12.

(a) Suppose $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$. Assume without loss of generality that $\mu(B) \le \mu(A)$. Since $\mu(C) \le \mu(B)$, by Lemma 7.12, $\mu(A)(1 - \varepsilon/2) < \mu(B)$ and thus

$$\mu(A) - \mu(B) \le \Big(\frac{1}{(1-\frac{\varepsilon}{2})} - 1\Big)\mu(B) \le \varepsilon\mu(B). \tag{7.2}$$

Since $\|\widetilde{x} - \widetilde{y}\| = \|\frac{x}{\mu(A)} - \frac{y}{\mu(B)}\| < \varepsilon$, we get that

$$\Big\|\frac{\mu(B)}{\mu(A)}x - y\Big\| < \varepsilon\mu(B). \tag{7.3}$$

Since $\|x - \frac{\mu(B)}{\mu(A)}x\| = \mu(A) - \mu(B)$, by (7.2) and (7.3) we get

$$\|x - y\| \le \Big\|x - \frac{\mu(B)}{\mu(A)}x\Big\| + \Big\|\frac{\mu(B)}{\mu(A)}x - y\Big\| \le 2\varepsilon\mu(B).$$

(b) Suppose now that $\|x - y\| < \varepsilon\|x\|$ (we no longer assume that $\mu(B) \le \mu(A)$). Then

$$\Big\|\frac{x}{\mu(A)} - \frac{y}{\mu(A)}\Big\| < \varepsilon.$$

Hence,

$$\Big\|\frac{y}{\mu(A)} - \frac{y}{\mu(B)}\Big\| = \Big|1 - \frac{\mu(B)}{\mu(A)}\Big| = \Big|\Big\|\frac{x}{\mu(A)}\Big\| - \Big\|\frac{y}{\mu(A)}\Big\|\Big| \le \Big\|\frac{x}{\mu(A)} - \frac{y}{\mu(A)}\Big\| < \varepsilon,$$

and thus,

$$\|\widetilde{x} - \widetilde{y}\| = \Big\|\frac{x}{\mu(A)} - \frac{y}{\mu(B)}\Big\| \le \Big\|\frac{x}{\mu(A)} - \frac{y}{\mu(A)}\Big\| + \Big\|\frac{y}{\mu(A)} - \frac{y}{\mu(B)}\Big\| < 2\varepsilon. \quad \square$$

Lemma 7.14. *Let $\mathcal{G}$ be a tree of signs, $\eta \in (0,4)$ and $\delta = \eta^2/16$. Given a δ-elementary $\mathcal{G}$-sign y, there exist $m \in \mathbb{N}$ and disjoint η-$\mathcal{G}$-functions $h_1, \dots, h_m$ such that* $\operatorname{supp} h_j \subseteq \operatorname{supp} y$ *for $j = 1, \dots, m$, and $\|\widetilde{y} - \widetilde{h}\| < \eta$, where $h = \sum_{j=1}^m h_j$.*

Proof. Let $B = \operatorname{supp} y$, and choose disjoint $\mathcal{G}$-functions $f_1, \dots, f_n$, such that

$$\|\widetilde{y} - \widetilde{f}\| < \delta\,, \tag{7.4}$$

where $f = \sum_{i=1}^{n} f_i$. Let $F_i = \operatorname{supp} f_i$, $F = \bigcup_{i=1}^{n} F_i$, $I = \{i \in \{1,\dots,n\} : \mu(F_i \cap B) > (1-\sqrt{\delta})\mu(F_i)\}$ and $J = \{1,\dots,n\} \setminus I$. We claim that

$$\sum_{i \in J} \mu(F_i) < \frac{\sqrt{\delta}}{2}\,\mu(F)\,. \tag{7.5}$$

Indeed, by Lemma 7.12 and (7.4)

$$\mu(F \cap B) \ge \mu(C) > \Big(1-\frac{\delta}{2}\Big)\max\{\mu(B),\mu(F)\} \ge \Big(1-\frac{\delta}{2}\Big)\mu(F)\,, \tag{7.6}$$

where $C = \{t \in [0,1] : y(t) = f(t) \neq 0\}$. Let $c = \mu(F) = \sum_{i=1}^{n} \mu(F_i)$ and $b = \sum_{i \in J} \mu(F_i)$. So, $c - b = \sum_{i \in I} \mu(F_i)$. By the definition of I we have that

$$\big(1-\sqrt{\delta}\big)\, b \ge \big(1-\sqrt{\delta}\big) \sum_{i \in J} \mu(F_i \cap B)\,. \tag{7.7}$$

By (7.6),

$$\sum_{i=1}^{n} \mu(F_i \cap B) > \Big(1-\frac{\delta}{2}\Big) \sum_{i=1}^{n} \mu(F_i) = \Big(1-\frac{\delta}{2}\Big) c\,. \tag{7.8}$$

Thus, by (7.7) and (7.8),

$$\big(1-\sqrt{\delta}\big)\, b + c - b \ge \sum_{i \in J} \mu(F_i \cap B) + \sum_{i \in I} \mu(F_i) \ge \sum_{i=1}^{n} \mu(F_i \cap B) > \Big(1-\frac{\delta}{2}\Big) c\,,$$

which gives $\frac{\sqrt{\delta}}{2} > b$, which proves (7.5).

Now for each $i \in I$, let $h_i = f_i \cdot \mathbf{1}_{F_i \cap B}$. By the definition of I, for every $i \in I$ we have

$$\|h_i - f_i\| = \mu(F_i \setminus G) < \sqrt{\delta}\mu(F_i) = \sqrt{\delta}\|f_i\|\,.$$

By Lemma 7.13(b), for every $i \in I$, h_i is an $(\eta/2)$-$\mathcal{G}$-function. Moreover,

$$\Big\|\sum_{i \in I} h_i - \sum_{i \in I} f_i\Big\| < \sqrt{\delta} \sum_{i \in I} \|f_i\| \le \sqrt{\delta}\|f\|\,.$$

Let $h = \sum_{i \in I} h_i$. By (7.5) we obtain that $\|h - f\| \le \frac{3\sqrt{\delta}}{2}\|f\|$. Therefore, again by Lemma 7.13(b), $\|\widetilde{h} - \widetilde{f}\| \le 3\sqrt{\delta}$. This yields that

$$\|\widetilde{y} - \widetilde{h}\| < \delta + 3\sqrt{\delta} < 4\sqrt{\delta} = \eta\,.$$

It is clear from the construction that $\operatorname{supp} h_i \subseteq \operatorname{supp} y$ for every $i \in I$. $\square$

Lemma 7.15. *Let $\mathcal{G}$ be a tree of signs, $\varepsilon \in (0,1)$ and $\delta = \varepsilon/9$. If $y_1, y_2, \dots$ is a disjoint finite or countable collection of δ-$\mathcal{G}$-functions then $y = \sum_i y_i$ is an ε-elementary $\mathcal{G}$-sign.*

Proof. By the definition of a tree of sets, if A and B are supports of members of $\mathcal{G}$ then either $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$. Hence, if $C_1, \dots, C_n$ are supports of members of $\mathcal{G}$ then there exists $I \subseteq \{1, \dots, n\}$ such that

$$\mu\Big(\bigcup_{i \in I} C_i\Big) = \mu\Big(\bigcup_{i=1}^{n} C_i\Big) \quad \text{and} \quad C_i \cap C_j = \emptyset, \quad \text{as} \quad i, j \in I \quad \text{and} \quad i \neq j\ . \tag{7.9}$$

If there are infinitely many y_is, we choose $n \in \mathbb{N}$ so that

$$\Big\| \widetilde{\sum_{i=1}^{n} y_i} - \widetilde{\sum_{i=1}^{\infty} y_i} \Big\| < \delta \tag{7.10}$$

(this is possible because $f_n \to f \neq 0$ in L_1, implies $\widetilde{f}_n \to \widetilde{f}$ in L_1). Now let $y' = \sum_{i=1}^{n} y_i$, $B_i = \operatorname{supp} y_i$ for all $i = 1, \dots, n$, and $a = \|y'\| = \sum_{i=1}^{n} \mu(B_i)$. For all $i = 1, \dots, n$, we choose $\mathcal{G}$-functions z_i with $\|\widetilde{z}_i - \widetilde{y}_i\| < \delta$ and put $C_i = \operatorname{supp} z_i$. By Lemma 7.12, for all $i = 1, \dots, n$,

$$\mu(C_i \cap B_i) > \Big(1 - \frac{\delta}{2}\Big)\mu(B_i)\ . \tag{7.11}$$

Hence,

$$\begin{aligned}\mu\Big(\bigcup_{i=1}^{n} C_i\Big) &\geq \mu\Big(\bigcup_{i=1}^{n} C_i \cap B_i\Big) = \sum_{i=1}^{n} \mu(C_i \cap B_i) \\ &> \sum_{i=1}^{n} \Big(1 - \frac{\delta}{2}\Big)\mu(B_i) = \Big(1 - \frac{\delta}{2}\Big)\, a\ .\end{aligned} \tag{7.12}$$

By Lemma 7.12, $(1 - \frac{\delta}{2})\mu(C_i) < \mu(B_i)$ for all $i = 1, \dots, n$, and hence

$$\sum_{i=1}^{n} \mu(C_i) < \frac{1}{1 - \frac{\delta}{2}} \sum_{i=1}^{n} \mu(B_i) \leq (1 + \delta)\, a\ . \tag{7.13}$$

Now we choose $I \subseteq \{1, \dots, n\}$ satisfying (7.9). By (7.9), (7.12) and (7.13) we have

$$\sum_{i \in J} \mu(C_i) < \left((1 + \delta) - \Big(1 - \frac{\delta}{2}\Big)\right) a = \frac{3}{2}\,\delta a\ , \tag{7.14}$$

where $J = \{1, \dots, n\} \setminus I$. By (7.11), for all $i = 1, \dots, n$,

$$\mu(B_i) < \Big(1 - \frac{\delta}{2}\Big)^{-1} \mu(C_i \cap B_i) \leq \Big(1 - \frac{\delta}{2}\Big)^{-1} \mu(C_i) \leq (1 + \delta)\mu(C_i)\ .$$

Hence, by (7.14) we have

$$\sum_{i\in J}\mu(B_i) < (1+\delta)\sum_{i\in J}\mu(C_i) < (1+\delta)\frac{3}{2}\,\delta a < 2\delta a\ . \tag{7.15}$$

By Lemma 7.13(a), for all $i = 1,\dots,n$,

$$\|z_i - y_i\| < 2\delta\min\{\mu(C_i),\mu(B_i)\} \le 2\delta\mu(B_i)\ .$$

Therefore, by (7.15), we have

$$\begin{aligned}\Big\|\sum_{i\in I} z_i - y'\Big\| &\le \sum_{i\in I}\|z_i - y_i\| + \sum_{i\in J}\|y_i\| \\ &< 2\delta\sum_{i\in I}\mu(B_i) + \sum_{i\in J}\mu(B_i) < 2\delta a + 2\delta a = 4\delta a\ .\end{aligned}$$

By Lemma 7.13(b), $\|\widetilde{\sum_{i\in I} z_i} - \widetilde{y}'\| < 8\delta$, and finally, by (7.10),

$$\Big\|\widetilde{\sum_{i\in I}} z_i - \widetilde{y}\Big\| < \Big\|\widetilde{\sum_{i\in I}} z_i - \widetilde{\sum_{i=1}^{n}} y_i\Big\| < 8\delta + \delta = \varepsilon\ .$$

□

Lemma 7.16. *Let $\mathcal{G}$ be a tree of signs. For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any tree of signs $\mathcal{G}$ and any sign $x \in L_1$ if x is a (finite or infinite) sum of disjoint δ-elementary $\mathcal{G}$-signs then x is an ε-elementary $\mathcal{G}$-sign.*

Proof. Let $\varepsilon > 0$, $\eta = \varepsilon/13$ and $\delta = \eta^2/16$. Assume $y_1, y_2, \dots$ are disjoint δ-elementary $\mathcal{G}$-signs and $x = \sum_i y_i$. By Lemma 7.14, for each i, we choose disjoint η-$\mathcal{G}$-functions $h_{i,j}$ with $\operatorname{supp} h_{i,j} \subseteq \operatorname{supp} y_i$ for all j, and

$$\Big\|\widetilde{y}_i - \widetilde{\sum_i h_{i,j}}\Big\| < \eta\ . \tag{7.16}$$

Since $(h_{i,j})_{i,j}$ are disjoint, by Lemma 7.15, there is an elementary $\mathcal{G}$-sign h with

$$\Big\|\widetilde{h} - \widetilde{\sum_{i,j} h_{i,j}}\Big\| < 9\eta\ . \tag{7.17}$$

By (7.16) and Lemma 7.13(a) ,

$$\Big\|y_i - \sum_j h_{i,j}\Big\| < 2\eta\|y_i\|\ ,$$

for all j, and hence

$$\Big\|\sum_i y_i - \sum_{i,j} h_{i,j}\Big\| < 2\eta \sum_i \|y_i\| = 2\eta\|y\| .$$

Again by Lemma 7.13(a),

$$\Big\|\widetilde{y} - \widetilde{\sum_{i,j} h_{i,j}}\Big\| < 4\eta . \tag{7.18}$$

Thus, by (7.17) and (7.18), y is a (13η)-elementary $\mathcal{G}$-sign. □

Lemma 7.17. *Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be trees of signs with $\mathcal{G}'$ related to $\mathcal{G}$ and $\mathcal{G}''$ related to $\mathcal{G}'$. Then $\mathcal{G}''$ is related to $\mathcal{G}$.*

Proof. Let $\mathcal{G}' = (g'_\alpha)_{\alpha\in\mathcal{D}}$ and $\mathcal{G}'' = (g''_\alpha)_{\alpha\in\mathcal{D}}$. Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon/2)$ from Lemma 7.16. Choose $k \in \mathbb{N}$ so that

$$g'_\alpha \quad \text{is a } \delta - \text{elementary } \mathcal{G} - \text{sign if} \quad |\alpha| \geq k . \tag{7.19}$$

Let $c' = \mu(\operatorname{supp}\mathcal{G}')$ and $c'' = \mu(\operatorname{supp}\mathcal{G}'')$, and observe that for every $\alpha \in \mathcal{D}$,

$$\|g'_\alpha\| = 2^{-|\alpha|}c' \quad \text{and} \quad \|g''_\alpha\| = 2^{-|\alpha|}c'' . \tag{7.20}$$

Choose $m \in \mathbb{N}$ so that

$$\frac{c'}{2^\ell} \leq \frac{(1+\varepsilon)\,c''}{2^m} \quad \text{implies} \quad \ell \geq k . \tag{7.21}$$

Let $n \geq m$, so that if $|\alpha| \geq n$ then g''_α is an $(\varepsilon/2)$-elementary $\mathcal{G}'$-sign. Fix any $\alpha \in \mathcal{D}$ with $|\alpha| \geq n$. We are going to show that g''_α is an ε-elementary $\mathcal{G}$-sign. To do this, we choose a finite sum $h' = \sum_{i=1}^s h'_i$ of disjoint $\mathcal{G}'$-functions h'_i so that

$$\|\widetilde{g}''_\alpha - \widetilde{h}'\| \leq \varepsilon/2 . \tag{7.22}$$

By Lemma 7.13, we have that $\|g''_\alpha - h'\| < \varepsilon\|g''_\alpha\|$ and hence, $\|h'\| < (1+\varepsilon)\|g''_\alpha\|$. Therefore, given any $i \in \{1,\dots,s\}$, by (7.20), we have that for a suitable m_i

$$\frac{c'}{2^{m_i}} = \|h'_i\| \leq \|h'\| < (1+\varepsilon)\|g''_\alpha\| \leq \frac{(1+\varepsilon)\,c''}{2^m} .$$

By (7.21), $m_i \geq k$ for all $i \leq s$, and by (7.19), h'_i is a δ-elementary $\mathcal{G}$-sign. By Lemma 7.16, h' is an $(\varepsilon/2)$-elementary $\mathcal{G}$-sign. Finally, by (7.22), g''_α is an ε-elementary $\mathcal{G}$-sign. □

Lemma 7.18. *Let $\emptyset \neq M \subseteq B_{L_\infty}$, $\mathcal{G}$ and $\mathcal{G}'$ be trees of signs with $\mathcal{G}'$ related to $\mathcal{G}$, and $b > 0$. If M b^+-norms $\mathcal{G}$, then M b^+-norms a piece of $\mathcal{G}'$.*

Proof. Assume $\mathcal{G}' = (g'_\alpha)_{\alpha\in\mathcal{D}}$. Choose $\varepsilon > 0$ so that M $(b + 2\varepsilon)$-norms $\mathcal{G}$. Let $\delta = \delta(\varepsilon)$ be as in Lemma 7.16 with respect to $\mathcal{G}$. Choose $k \in \mathbb{N}$ so that g'_β is a δ-elementary $\mathcal{G}$-sign if $|\beta| \geq k$. Fix any $\beta \in \mathcal{D}$ with $|\beta| \geq k$ and set $h''_\alpha = h'_{\beta\,\alpha}$ for all $\alpha \in \mathcal{D}$. Thus, $\mathcal{G}'' = (g''_\alpha)_{\alpha\in\mathcal{D}}$ is a piece of $\mathcal{G}'$. Now suppose x is an elementary $\mathcal{G}''$-sign, that is, $x = \sum_{i=1}^m \varepsilon_i g'_{\beta\,\alpha_i}$ for some $m \in \mathbb{N}$, $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathcal{D}$. Since $|\beta\,\alpha_i| \geq k$, we have that $g'_{\beta\,\alpha_i}$ is a δ-elementary $\mathcal{G}$-sign, and so is $\varepsilon_i g'_{\beta\,\alpha_i}$ for each $i = 1, \dots, m$. By Lemma 7.16, x is an ε-elementary $\mathcal{G}$-sign. Hence, there is an elementary $\mathcal{G}$-sign y with $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$. Since $M(\widetilde{x}) \geq b + 2\varepsilon$, we get that $M(\widetilde{y}) \geq b + \varepsilon$. □

Definition 7.19 (Rosenthal [124]). We say that a sequence $(A_n, B_n)_{n=1}^\infty$ of pairs of subsets of a set S with $A_n \cap B_n = \emptyset$ for all $n \in \mathbb{N}$, is *Boolean independent*, if for any two disjoint finite sets of indices $I, J \subset \mathbb{N}$ we have

$$\bigcap_{i\in I} A_i \cap \bigcap_{j\in J} B_j \neq \emptyset\,,$$

(under the convention that $\bigcap_{i\in\emptyset} C_i = S$).

Lemma 7.20. *Let S be a set, (f_n) a uniformly bounded sequence of real-valued functions defined on S, and $0 < a < b < \infty$. For each $n \in \mathbb{N}$, let $A_n = \{s \in S : |f_n(s)| > b\}$ and $B_n = \{s \in S : |f_n(s)| < a\}$. Assume that the sequence (A_n, B_n) is Boolean independent. Then there exists a subsequence (f'_n) of (f_n) such that for all $n \in \mathbb{N}$, and all real scalars $(c_j)_{j=1}^n$, we have*

$$\sup_{s\in S}\Big|\sum_{i=1}^n c_i f'_i(s)\Big| \geq \frac{b-a}{2}\sum_{i=1}^n |c_i|\,. \tag{7.23}$$

Lemma 7.20 has a somewhat combinatorial nature. Its proof is logically divided into several steps (sublemmas below) first of which (Sublemma 1) is contained in Rosenthal's paper [124]. The rest of the proof was communicated to us by Mykhaylyuk.

Sublemma 1. *Let S be a set, (f_n) a sequence of real-valued functions defined on S, $r \in \mathbb{R}$ and $\delta > 0$. For each $n \in \mathbb{N}$, set $F_n = \{s \in S : f_n(s) > r + \delta\}$ and $G_n = \{s \in S : f_n(s) < r\}$. Assume that the sequence (F_n, G_n) is Boolean independent. Then for all $n \in \mathbb{N}$ and all real scalars $(c_j)_{j=1}^n$ we have*

$$\sup_{s\in S}\Big|\sum_{i=1}^n c_i f_i(s)\Big| \geq \frac{\delta}{2}\sum_{i=1}^n |c_i|\,.$$

Proof. Fix n and real scalars $(c_j)_{j=1}^n$. Let $I = \{i \in \{1, \dots, n\} : c_i > 0\}$ and $J = \{j \in \{1, \dots, n\} : c_j \leq 0\}$. By the Boolean independence of (F_n, G_n), there exists $s_1 \in \bigcap_{i\in I} A_i \cap \bigcap_{j\in J} B_j$ and $s_2 \in \bigcap_{i\in I} B_i \cap \bigcap_{j\in J} A_j$. Then

(a) $\sum_{i\in I} c_i f_i(s_1) \ge (r+\delta)\sum_{i\in I} |c_i|$;

(b) $\sum_{j\in J} c_j f_j(s_1) = \sum_{j\in J} -c_j(-f_j(s_1)) \ge -r\sum_{j\in J} |c_j|$;

(c) $-\sum_{i\in I} c_i f_i(s_2) = \sum_{i\in I} c_i(-f_i(s_2)) \ge -r\sum_{i\in I} |c_i|$;

(d) $-\sum_{j\in J} c_j f_j(s_2) \ge (r+\delta)\sum_{j\in J} |c_j|$.

Adding (a), (b), (c) and (d), we obtain

$$\sum_{i=1}^{n} c_i f_i(s_1) - \sum_{i=1}^{n} c_i f_i(s_2) \ge \delta \sum_{i=1}^{n} |c_i| \,.$$

Hence,

$$\begin{aligned}\sup_{s\in S}\Big|\sum_{i=1}^{n} c_i f_i(s)\Big| &\ge \max\Big\{\Big|\sum_{i=1}^{n} c_i f_i(s_1)\Big|, \Big|\sum_{i=1}^{n} c_i x_i(s_2)\Big|\Big\} \\ &\ge \frac{1}{2}\Big(\sum_{i=1}^{n} c_i f_i(s_1) - \sum_{i=1}^{n} c_i x_i(s_2)\Big) > \frac{\delta}{2}\sum_{i=1}^{n} |c_i|.\end{aligned}$$

□

For the remaining sublemmas we need to consider a "localized" version of Boolean independence. Denote by $\mathbb{N}^{<\omega}$ the set of all finite subsets of $\mathbb{N}$.

Let $\alpha = ((A_n, B_n))_{n\in\mathbb{N}}$ be a sequence of pairs of subsets of a set S with $A_n \cap B_n = \emptyset$ for each $n \in \mathbb{N}$, and let $I, J \in \mathbb{N}^{<\omega}$ with $I \cap J = \emptyset$. We set

$$C_\alpha(I,J) = \bigcap_{i\in I} A_i \cap \bigcap_{j\in J} B_j \;.$$

We say that a sequence α is *Boolean independent on a subset* $T \subseteq S$ if $C_\alpha(I,J) \cap T \neq \emptyset$ for each $I, J \in \mathbb{N}^{<\omega}$ with $I \cap J = \emptyset$.

The following fact follows directly from the definition.

Sublemma 2. *Let a sequence $\alpha = ((A_n, B_n))_{n\in\mathbb{N}}$ be Boolean independent on $T \subseteq S$ and $I, J \in \mathbb{N}^{<\omega}$ with $I \cap J = \emptyset$. Then the sequence $\alpha' = ((A_n, B_n))_{n\in\mathbb{N}\setminus(I\cup J)}$ is Boolean independent on $T \cap C_\alpha(I,J)$.*

Sublemma 3. *Let a sequence $\alpha = ((A_n, B_n))_{n\in\mathbb{N}}$ be Boolean independent on $T \subseteq S$, $(n_k)_{k=1}^\infty$ be a strictly increasing sequence of the integers, and $(I_k)_{k=1}^\infty$ and $(J_k)_{k=1}^\infty$ be sequences in $\mathbb{N}^{<\omega}$ with $I_k, J_k \subseteq \mathbb{N} \cap (n_k, n_{k+1})$ and $I_k \cap J_k = \emptyset$. Then the sequence $\alpha' = ((A'_k, B'_k))_{k\in\mathbb{N}}$, defined by $A'_k = A_{n_k} \cap C_\alpha(I_k, J_k)$ and $B'_k = B_{n_k}$ is Boolean independent on T as well.*

Proof. Let $I, J \in \mathbb{N}^{<\omega}$ with $I \cap J = \emptyset$. Then

$$\begin{aligned} C_{\alpha'}(I,J) &= \bigcap_{k\in I}\big(A_{n_k} \cap C_\alpha(I_k,J_k)\big) \cap \bigcap_{\ell\in J}\big(B_{n_\ell} \cap C_\alpha(I_\ell,J_\ell)\big) \\ &= \bigcap_{k\in I}\big(A_{n_k} \cap \bigcap_{i\in I_k} A_i \cap \bigcap_{j\in J_k} B_j\big) \cap \bigcap_{\ell\in J}\big(B_{n_\ell} \cap \bigcap_{i\in I_\ell} A_i \cap \bigcap_{j\in J_\ell} B_j\big) \\ &= C_\alpha(I^*,J^*)\,, \end{aligned}$$

where

$$I^* = \bigcup_{k\in I}\big(\{n_k\} \cup I_k\big) \cup \bigcup_{\ell\in J} I_\ell \ \text{ and } \ J^* = \bigcup_{k\in I} J_k \cup \bigcup_{\ell\in J}\big(\{n_\ell\} \cup J_\ell\big)\,.$$

Hence, $C_{\alpha'}(I,J) \cap T = C_\alpha(I^*,J^*) \cap T \neq \emptyset$. □

Sublemma 4. *Let a sequence $\alpha = ((A_n, B_n))_{n\in\mathbb{N}}$ be Boolean independent on $T \subseteq S$ and $A_n = U_n \cup V_n$ for each $n \in \mathbb{N}$. Then there exist a strictly increasing sequence of the integers $(n_k)_{k=1}^\infty$ and a sequence $(A'_k)_{k=1}^\infty$ of sets $A'_k \in \{U_{n_k}, V_{n_k}\}$, such that the sequence $\alpha' = ((A'_k, B'_k))_{k\in\mathbb{N}}$, where $B'_k = B_{n_k}$ is Boolean independent on T as well.*

Proof. First we consider the following partial case.

Assumption 1. *For every $m \in \mathbb{N}$ there are $n = n(m) > m$, $A = A(m) \in \{U_n, V_n\}$ and finite sets $I = I(m), J = J(m) \subset \mathbb{N} \cap (n, +\infty)$ with $I \cap J = \emptyset$ such that $(T \cap A) \cap C_\alpha(I,J) = \emptyset$.*

Under this assumption, we construct recursively strictly increasing sequences of integers $(m_k)_{k=1}^\infty$ and $(n_k)_{k=1}^\infty$ such that

$$\begin{aligned} 1 = m_1 < n_1 = n(m_1) < m_2 = \max\big(I(m_1) \cup J(m_1)\big) < n_2 = n(m_2) < m_3 \\ = \max\big(I(m_2) \cup J(m_2)\big) < \dots\,. \end{aligned}$$

For each $k \in \mathbb{N}$, let $I_k = I(m_k)$, $J_k = J(m_k)$, $A'_k = U_{n_k}$ if $A(m_k) = V_{n_k}$ and $A'_k = V_{n_k}$ if $A(m_k) = U_{n_k}$. Observe that by Assumption 1,

$$P_k = T \cap A'_k \cap C_\alpha(I_k,J_k) = T \cap A_{n_k} \cap C_\alpha(I_k,J_k)$$

for each $k \in \mathbb{N}$. By Sublemma 3, the sequence $((P_k, B'_k))_{k\in\mathbb{N}}$ is Boolean independent on T. Hence, the sequence $((A'_k, B'_k))_{k\in\mathbb{N}}$ is Boolean independent on T as well by the inclusion $P_k \subseteq A'_k$.

Our next case is a little bit more general than the previous one.

Assumption 2. *There exist a set $T^* \subseteq T$, and a subsequence $\beta = ((A_{i_n}, B_{i_n}))_{n\in\mathbb{N}}$ of α, Boolean independent on T^*, such that for every $m \in \mathbb{N}$, there are $n > m$ and $A \in \{U_{i_n}, V_{i_n}\}$ such that the sequence $((A_{i_k}, B_{i_k}))_{k>n}$ is not Boolean independent on $A \cap T^*$.*

Observe that Assumption 2 implies that Assumption 1 is satisfied for β instead of α and T^* instead of T. Since Sublemma 4 asserts the existence of subsequences, it then will do for α itself. And, since the independence on T^* implies the independence on T, we conclude that the sublemma is proved under Assumption 2.

The last assumption is the negation of Assumption 2, which will complete the proof.

Assumption 3. *Let Assumption 2 be false, that is, for any $T^* \subseteq T$ and any subsequence $\beta = ((A_{i_n}, B_{i_n}))_{n\in\mathbb{N}}$ of α, Boolean independent on T^*, there exists $m \in \mathbb{N}$ such that for every $n > m$ and $A \in \{U_{i_n}, V_{i_n}\}$ the sequence $((A_{i_k}, B_{i_k}))_{k>n}$ is Boolean independent on $A \cap T^*$.*

By induction on k we construct a strictly increasing sequence of integers $(n_k)_{k=1}^{\infty}$ and a sequence of sets $(A'_k)_{k=1}^{\infty}$ with $A'_k \in \{U_{n_k}, V_{n_k}\}$ such that for each $k \in \mathbb{N}$ the sequence $\alpha_k = ((P_i^{(k)}, Q_i^{(k)}))_{i\in\mathbb{N}}$ of pairs

$$\big(P_i^{(k)}, Q_i^{(k)}\big) = \begin{cases} (A'_i, B'_i), & i \le k, \\ (A_{i+n_k-k}, B_{i+n_k-k}), & i > k, \end{cases}$$

is Boolean independent on T^*.

By Assumption 3 applied to $\beta = \alpha$ and $T^* = T$, there exist $n_1 \in \mathbb{N}$ and a set $A'_1 \in \{U_{n_1}, V_{n_1}\}$ so that the sequence $((A_n, B_n))_{n>n_1}$ is Boolean independent on $A'_1 \cap T$. Then the sequence

$$\alpha_1 = \big((A'_1, B'_1), (A_{n_1+1}, B_{n_1+1}), (A_{n_1+2}, B_{n_1+2}), \dots\big)$$

is Boolean independent on T as well.

Assume that integers $(n_i)_{i=1}^{k}$ and sets $(A'_i)_{i=1}^{k}$ have been chosen so that the sequences $\alpha_1, \alpha_2, \dots, \alpha_k$ are Boolean independent on T. Let $\mathcal{T}$ be the collection of all sets $T^* = (\bigcap_{i\in I} A'_i) \cap (\bigcap_{j\in J} B'_j)$, where $\{1, 2, \dots, k\} = I \sqcup J$. Observe that $\mathcal{T}$ is finite, and hence by Sublemma 1, the sequence $\beta_k = ((A_n, B_n))_{n>n_k}$ is Boolean independent on every set $T^* \in \mathcal{T}$. Applying Assumption 3 to β_k and each set $T^* \in \mathcal{T}$, we find a number $n_{k+1} > n_k$ such that the sequence $\beta_{k+1} = ((A_n, B_n))_{n>n_{k+1}}$ is Boolean independent on each set $A \cap T^*$ where $A \in \{U_{n_{k+1}}, V_{n_{k+1}}\}$ and each $T^* \in \mathcal{T}$. We choose an arbitrary $A'_{k+1} \in \{U_{n_{k+1}}, V_{n_{k+1}}\}$. Notice that by Sublemma 1, the sequence β_{k+1} is Boolean independent on any set $B_{n_{k+1}} \cap T^*$, where $T^* \in \mathcal{T}$. Since $B'_{k+1} = B_{n_{k+1}}$, we deduce that the sequence

$$\alpha_{k+1} = \big((A'_1, B'_1), \dots, (A'_{k+1}, B'_{k+1}), (A_{n_{k+1}+1}, B_{n_{k+1}+1}), \\ (A_{n_{k+1}+2}, B_{n_{k+1}+2}), \dots\big)$$

is Boolean independent on T.

Now observe that for any $I, J \in \mathbb{N}^{<\omega}$ with $I \cap J = \emptyset$ and $\max(I \cup I) \le k$ we have $C'_\alpha(I, J) = C_{\alpha_k}(I, J)$. Thus, the sequence α' is Boolean independent on S. $\square$

Proof of Lemma 7.20. Let $U_n = \{s \in A_n : f_n(s) \geq 0\}$ and $V_n = \{s \in A_n : f_n(s) < 0\}$, and choose the corresponding sequences (n_k) and (A'_k) by Sublemma 4. Denote $N_1 = \{k \in \mathbb{N} : A'_k = U_{n_k}\}$ and $N_2 = \{k \in \mathbb{N} : A'_k = V_{n_k}\}$. Since $N_2 \cup N_2 = \mathbb{N}$, at least one of the sets N_1, N_2 is infinite.

If N_1 is infinite, we use Sublemma 1 for $r = a$, $\delta = b - a$ and the sequence $((F_{n_k}, G_{n_k}))_{k \in N_1}$ to prove the lemma (it is Boolean independent, because $A'_k = U_{n_k} \subseteq F_{n_k}$ and $B_{n_k} \subseteq G_{n_k}$ for each $k \in N_1$).

If N_2 is infinite, we use Sublemma 1 for $r = -b$, $\delta = b - a$ and the sequence $((F_{n_k}, G_{n_k}))_{k \in N_2}$ to prove the lemma (it is Boolean independent, because $A'_k = V_{n_k} \subseteq G_{n_k}$ and $B_{n_k} \subseteq F_{n_k}$ for each $k \in N_2$). □

Lemma 7.21. *Let $\mathcal{G} = (g_\alpha)_{\alpha \in \mathcal{D}}$ be a tree of signs and $G_\alpha = \operatorname{supp} g_\alpha$, for each $\alpha \in \mathcal{D}$. Let E be a set of positive measure belonging to the σ-algebra $\Sigma_{\mathcal{G}}$ generated by $(G_\alpha)_{\alpha \in \mathcal{D}}$. Then for every $\delta > 0$, there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ with* $\operatorname{supp} \mathcal{G}' = E$ *such that every member of $\mathcal{G}'$ is a δ-elementary $\mathcal{G}$-sign.*

Proof. Observe that for every $\varepsilon > 0$ and every $F \in \Sigma^+_{\mathcal{G}}$, there exists an ε-elementary $\mathcal{G}$-sign x with $\operatorname{supp} x = F$. Indeed, let $\alpha_1, \ldots, \alpha_k \in \mathcal{D}$ be so that $\mu(F \Delta \bigsqcup_{i=1}^k G_{\alpha_i}) < \varepsilon/2\mu(F)$, the proof is the same as for the case of the dyadic tree, when the G_{α_i}s are disjoint dyadic intervals. Define $H = F \setminus \bigcup_{i=1}^k G_{\alpha_i}$ and $g = \mathbf{1}_F \sum_{i=1}^k g_{\alpha_i} + \mathbf{1}_H$. Then

$$\Big\| g - \sum_{i=1}^k g_{\alpha_i} \Big\| = \mu\Big(F \Delta \bigsqcup_{i=1}^k G_{\alpha_i}\Big) < \frac{\varepsilon}{2}\mu(F),$$

and thus $\|\widetilde{g - \sum_{i=1}^k g_{\alpha_i}}\| < 2(\varepsilon/2) = \varepsilon$, by Lemma 7.13(b).

We choose recursive sets $E_\alpha \in \Sigma_{\mathcal{G}}$, for every $\alpha \in \mathcal{D}$, so that $E_\emptyset = E$ and $E_\alpha = E_{\alpha\,0} \sqcup E_{\alpha\,1}$ with $\mu(E_{\alpha\,0}) = \mu(E_{\alpha\,1})$. By our initial observation, for each $\alpha \in \mathcal{D}$ we choose an ε_α-elementary $\mathcal{G}$-sign g'_α with $\operatorname{supp} g'_\alpha = E_\alpha$, where $\varepsilon_\alpha = \min\{\delta, 2^{-|\alpha|}\}$. Thus, $\mathcal{G}' = (g'_\alpha)_{\alpha \in \mathcal{D}}$ is the desired tree. □

Lemma 7.22. *Let $\mathcal{G} = (g_\alpha)_{\alpha \in \mathcal{D}}$ be a tree of signs and $M \subseteq B_{L_\infty}$. Let $b, \varepsilon > 0$ be so that M $(b+\varepsilon)$-norms every elementary $\mathcal{G}$-sign x with* $\operatorname{supp} x = \operatorname{supp} \mathcal{G}$. *Then there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ such that M b^+-norms $\mathcal{G}'$.*

Proof. We set $E = \operatorname{supp} \mathcal{G}$ and observe that

$$\begin{aligned} &\text{If } x \text{ is an elementary } \mathcal{G}\text{-sign with } \mu(\operatorname{supp} x) \geq \Big(1 - \frac{\varepsilon}{2}\Big)\mu(E) \\ &\text{then } M(\widetilde{x}) \geq b + \frac{\varepsilon}{2}. \end{aligned} \tag{7.24}$$

Indeed, denote $A = \operatorname{supp} x$ and choose an elementary $\mathcal{G}$-sign y with $\operatorname{supp} y = E \setminus A$. Since $\operatorname{supp}(x + y) = E$, by the lemma assumption, $M(x + y) \geq (b + \varepsilon)\mu(E)$.

Suppose $\mu(\operatorname{supp} x) \geq (1 - \frac{\varepsilon}{2})\mu(E)$. Then $M(y) \leq \|y\| \leq \frac{\varepsilon}{2}\,\mu(E)$, and hence

$$M(\widetilde{x}) = \frac{M(x)}{\mu(E)} \geq \frac{M(x+y) - M(y)}{\mu(E)} \geq b + \varepsilon - \frac{\varepsilon}{2} = b + \frac{\varepsilon}{2}\,.$$

Let $\delta = \delta(\varepsilon/4)$, be from the assertion of Lemma 7.16, and let $\Sigma_{\mathcal{G}}$ be the σ-algebra generated by $\mathcal{G}$ (that is, generated by the supports of elements of $\mathcal{G}$). We are going to prove that there is $F \in \Sigma_{\mathcal{G}}^+$ such that M $(b + \varepsilon/4)$-norms every δ-elementary $\mathcal{G}$-sign x with $\operatorname{supp} x \subseteq F$.

Suppose there is no such F. Denote by $\mathcal{M}$ the set of all finite or countable disjoint (unordered) collections $(x_i)_{i\in I}$ of δ-elementary $\mathcal{G}$-signs such that $\operatorname{supp} x_i \in \Sigma_{\mathcal{G}}$ and $M(x_i) < (b + \varepsilon/4)\|x_i\|$ for each $i \in I$. We endow $\mathcal{M}$ with the inclusion ordering, that is, $(x_i)_{i\in I} \leq (y_j)_{j\in J}$ provided for each $i \in I$ there exists $j \in J$ such that $x_i = y_j$. Let $\mathcal{L} \subseteq \mathcal{M}$ be a chain. Setting $X = \bigcup_{Y\in\mathcal{L}} Y$ we obtain that $X \in \mathcal{M}$. Indeed, obviously X is a disjoint collections of δ-elementary $\mathcal{G}$-signs. It is, at most countable, because $\sum_{x\in X} \mu(\operatorname{supp} x) \leq \mu(F)$. Since $Y \leq X$ for each $Y \in \mathcal{L}$, X is an upper bound of $\mathcal{L}$ in $\mathcal{M}$. By Zorn's lemma, $\mathcal{M}$ has a maximal element $Y = (y_i)_{i\in I}$. We have that $\bigcup_{i\in I} \operatorname{supp} y_i = E$, because otherwise, by the assumption, we would obtain a contradiction with the maximality of Y. Thus, setting $y = \sum_{i\in I} y_i$, we obtain that $\operatorname{supp} y = E$ and

$$M(y) \leq \sum_{i\in I} M(y_i) < \sum_{i\in I}\Big(b + \frac{\varepsilon}{4}\Big)\|y_i\| = \Big(b + \frac{\varepsilon}{4}\Big)\mu(E)\,. \tag{7.25}$$

By Lemma 7.16, y is an $(\varepsilon/4)$-elementary $\mathcal{G}$-sign, that is, there exists an elementary $\mathcal{G}$-sign x with $\|\widetilde{x} - \widetilde{y}\| < \varepsilon/4$. Hence, by (7.25),

$$M(\widetilde{x}) \leq M(\widetilde{y}) + M(\widetilde{y} - \widetilde{x}) \leq \frac{M(y)}{\mu(E)} + \|\widetilde{x} - \widetilde{y}\| < b + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = b + \frac{\varepsilon}{2}\,, \tag{7.26}$$

and by Lemma 7.13(a), $\|x - y\| < \frac{\varepsilon}{2}\|y\| = \frac{\varepsilon}{2}\mu(E)$, therefore

$$\mu(\operatorname{supp} x) = \|x\| \geq \|y\| - \|x - y\| > \Big(1 - \frac{\varepsilon}{2}\Big)\mu(E)\,. \tag{7.27}$$

Now (7.26) together with (7.27) contradicts (7.24), proving the existence of $F \in \Sigma_{\mathcal{G}}^+$ such that M $(b + \frac{\varepsilon}{4})$-norms every δ-elementary $\mathcal{G}$-sign x with $\operatorname{supp} x \subseteq F$.

By Lemma 7.16, there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ with $\operatorname{supp} \mathcal{G}' = F$ such that every member of $\mathcal{G}'$ is a δ-elementary $\mathcal{G}$-sign. Thus, M $(b + \frac{\varepsilon}{4})$-norms $\mathcal{G}'$. In particular, M b^+-norms $\mathcal{G}'$. □

Lemma 7.23. *Let $M \subseteq B_{L_\infty}$ b^+-norm a tree of signs $\mathcal{G}$, and assume $M = \bigcup_{i=1}^k M_i$. Then there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$, and an index $i \in \{1,\dots,k\}$ so that M_i b^+-norm $\mathcal{G}'$.*

Proof. By transitivity of the relationship (Lemma 7.17), it suffices to consider the case of $k = 2$. In this case, we choose $\varepsilon > 0$ so that M $(b + \varepsilon)$-norms $\mathcal{G}$. Let $\mathcal{G} = (g_\alpha)_{\alpha\in\mathcal{D}}$ and $G_\alpha = \operatorname{supp} g_\alpha$ for each $\alpha \in \mathcal{D}$. We consider the following cases.

(a) *There is $\alpha \in \mathcal{D}$ such that M_1 $(b + \frac{\varepsilon}{2})$-norms all elementary $\mathcal{G}$-signs x with* $\operatorname{supp} x = G_\alpha$. In this case we apply Lemma 7.22.

(b) *For each $\alpha \in \mathcal{D}$ there exists an elementary $\mathcal{G}$-sign g'_α with $M_1(\widetilde{g'_\alpha}) < b + \frac{\varepsilon}{2}$ and* $\operatorname{supp} g'_\alpha = G_\alpha$. We claim that in this case M_2 $(b + \varepsilon)$-norms $\mathcal{G}' = (g'_\alpha)_{\alpha\in\mathcal{D}}$. Indeed, if $x = \sum_{i=1}^m \epsilon_i g_{\alpha_i}$ is an elementary $\mathcal{G}'$-sign, $\epsilon_i = \pm 1$ then

$$M_1(x) \le \sum_{i=1}^m M_1(g_{\alpha_i}) < \sum_{i=1}^m \bigl(b + \frac{\varepsilon}{2}\bigr)\|g_{\alpha_i}\| = \bigl(b + \frac{\varepsilon}{2}\bigr)\|x\|.$$

Since $(M_1 \cup M_2)(\widetilde{x}) \ge b + \varepsilon$ and $M_1(\widetilde{x}) < b + \varepsilon$, one gets that $M_2(\widetilde{x}) \ge b + \varepsilon$. It remains to observe that $\mathcal{G}'$ is related to $\mathcal{G}$. □

Lemma 7.24. *Let $M \subseteq B_{L_\infty}$ b^+-norm a tree of signs $\mathcal{G}$. Then there exists $\eta > b$ such that for every $\varepsilon > 0$, there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ such that*

$$|M(\widetilde{x}) - \eta| < \varepsilon \text{ for every elementary } \mathcal{G}'\text{-sign } x\,. \tag{7.28}$$

Proof. Let $\mathcal{G} = (g_\alpha)_{\alpha\in\mathcal{D}}$. For each $\alpha \in \mathcal{D}$, we set $G_\alpha = \operatorname{supp} g_\alpha$ and $\eta = \sup_{\alpha\in\mathcal{D}} \eta_\alpha$, where

$$\eta_\alpha = \inf\bigl\{M(\widetilde{x}) :\ x \text{ is an elementary } \mathcal{G}'\text{-sign with } \operatorname{supp} x = G_\alpha\bigr\}\,.$$

Since M b^+-norms $\mathcal{G}$, we have that $\eta > b$. Let $\varepsilon > 0$, $\tau = \varepsilon/2$ and $\alpha_0 \in \mathcal{D}$ be such that

$$\eta_{\alpha_0} > \eta - \tau\,. \tag{7.29}$$

For each $\alpha \ge \alpha_0$ we choose an elementary $\mathcal{G}$-sign x_α with $\operatorname{supp} x_\alpha = G_\alpha$ and

$$M(\widetilde{x_\alpha}) < \eta_\alpha + \tau \le \eta + \tau\,. \tag{7.30}$$

Define a tree of signs $\mathcal{H} = (x_\alpha)_{\alpha\in\mathcal{D}}$. Then (7.30) implies that

$$|M(\widetilde{x}) - \eta| \le \eta + \tau \text{ for every elementary } \mathcal{H}\text{-sign } x\,. \tag{7.31}$$

It follows from (7.29) that $M(\widetilde{x}) \ge \eta - \tau = \eta - \varepsilon + \tau$ for every elementary $\mathcal{H}$-sign x with $\operatorname{supp} x = \operatorname{supp} \mathcal{H} = \operatorname{supp} H_{\alpha_0}$. By Lemma 7.22, there exists a tree of signs $\mathcal{G}'' = (g''_\alpha)_{\alpha\in\mathcal{D}}$ related to $\mathcal{H}$ (and hence, to $\mathcal{G}$) such that M $(\eta-\varepsilon)^+$-norms $\mathcal{G}''$.

Let $\delta = \delta(\tau)$ be from Lemma 7.16. Using Definition 7.9, we choose $k \in \mathbb{N}$ so that g''_α is a δ-elementary $\mathcal{H}$-sign for each $\alpha \in \mathcal{D}$ with $|\alpha| \ge k$. Fix any $\alpha_1 \in \mathcal{D}$ with $|\alpha_1| = k$ and set $g'_\alpha = g''_{\alpha_1\alpha}$ for each $\alpha \in \mathcal{D}$. Then for the tree of signs

$\mathcal{G}' = (g'_\alpha)_{\alpha\in\mathcal{D}}$ we have that for every $\alpha \in \mathcal{D}$, g'_α is a δ-elementary $\mathcal{H}$-sign. Being a piece of $\mathcal{G}''$, $\mathcal{G}'$ is related to $\mathcal{H}$ and to $\mathcal{G}$ as well. By Lemma 7.16, every elementary $\mathcal{G}'$-sign is a τ-elementary $\mathcal{H}$-sign. By (7.31), $|M(\widetilde{x}) - \eta| < \eta + 2\tau = \eta + \varepsilon$ for every τ-elementary $\mathcal{G}'$-sign x. □

Lemma 7.25. *Suppose that sets $M_1, \dots, M_k \subseteq B_{L_\infty}$ all b^+-norm a tree of signs $\mathcal{G}$. Then there exists $\tau \in (0,1)$ such that for every $\varepsilon > 0$, there exist a tree $\mathcal{G}'$ related to $\mathcal{G}$ and numbers $\eta_j \geq b + \tau$ for each $j = 1, \dots, k$, such that*

$$|M_j(\widetilde{x}) - \eta_j| < \varepsilon \text{ for every elementary } \mathcal{G}'\text{-sign } x . \tag{7.32}$$

Proof. Let $\beta \in (0,2)$ so that M_j $(b+\beta)$-norms $\mathcal{G}$ for every $j = 1, \dots, k$, set $\tau = \beta/2$ and fix any $\varepsilon > 0$. We choose inductively trees of signs $\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_k$ and numbers $\eta_1, \dots, \eta_k$ so that for every $j = 1, \dots, k$, $\eta_j > b + \beta/2$, the tree $\mathcal{G}_j$ is related to $\mathcal{G}_{j-1}$ and

$$|M_j(\widetilde{x}) - \eta_j| < \varepsilon \text{ for every elementary } \mathcal{G}_j\text{-sign } x . \tag{7.33}$$

To see that this is possible, fix $i \in \{0, \dots, k-1\}$ and suppose that $\mathcal{G}_i$ and η_i has been chosen. By Lemma 7.17, $\mathcal{G}_i$ is related to $\mathcal{G}$, and, by Lemma 7.18, we choose a piece $\mathcal{G}'_i$ of $\mathcal{G}_i$ such that M_{i+1} $(b+\beta/2)$-norms $\mathcal{G}'_i$. By Lemma 7.24, we choose a tree of signs $\mathcal{G}_{i+1}$ related to $\mathcal{G}_i$ and η_i with the desired properties.

The induction completed, we have that, by Lemma 7.17, $\mathcal{G}_k$ is related to all $\mathcal{G}_i$s. By Lemma 7.16, we choose a piece $\mathcal{G}'$ of $\mathcal{G}_k$ so that every elementary $\mathcal{G}'$-sign is an $(\varepsilon/2)$-elementary $\mathcal{G}_j$ function for all $j = 1, \dots, k$. Equation (7.32) now follows from this and (7.33). □

Lemma 7.26. *Let $s \in \mathbb{N}$, $\tau \in (0,1)$, and $\varepsilon = \tau/(2(4s+1))$. Suppose that a tree of signs $\mathcal{G}$ and sets $M_1, \dots, M_s \subseteq B_{L_\infty}$ satisfy the following hypothesis:*

For every $j = 1, \dots, k$, there exists $\eta_j \geq b + \tau$ such that for every elementary $\mathcal{G}$-sign x we have

(a) $M_j(\widetilde{x}) < \eta_j + \varepsilon$;

(b) $M_j(\widetilde{x}) > \eta_j - \varepsilon$ *if* $\operatorname{supp} x = \operatorname{supp} \mathcal{G}$.

Then there exists a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ such that each M_j b^+-norms $\mathcal{G}'$ for $j = 1, \dots, s$.

Proof. Without loss of generality, for simplicity of the notation, we assume that $\operatorname{supp} \mathcal{G} = [0,1]$. Let $\gamma = 1/(2s)$. We claim that for each $j \in \{1, \dots, s\}$

$$M_j(\widetilde{x}) > \eta_j - \frac{2\varepsilon}{\gamma}, \text{ provided } x \text{ is an elementary } \mathcal{G}\text{-sign with } \mu(\operatorname{supp} x) > \gamma . \tag{7.34}$$

Indeed, let x be an elementary $\mathcal{G}$-sign with $\lambda = \mu(\operatorname{supp} x) > \gamma$. Choose an elementary $\mathcal{G}$-sign y with support disjoint from x and $\|x\| + \|y\| = 1$. Applying (b) to $z = x + y = \widetilde{z}$, we obtain

$$\eta_j - \varepsilon < M_j(x) + M_j(y) = \lambda M_j(\widetilde{x}) + (1-\lambda)M_j(\widetilde{y}) \overset{\text{by (a)}}{\leq} \lambda M_j(\widetilde{x}) + (1-\lambda)(\eta_j + \varepsilon)\,.$$

Thus,

$$\eta_j - \varepsilon - (1-\lambda)(\eta_j + \varepsilon) < \lambda M_j(\widetilde{x})\,. \tag{7.35}$$

Since the left-hand side of (7.35) equals $\lambda(\eta_j + \varepsilon) - 2\varepsilon$, we obtain $M_j(\widetilde{x}) > \eta_j + \varepsilon - 2\varepsilon/\lambda > \varepsilon - 2\varepsilon/\gamma$, proving (7.34).

Let $\Sigma_{\mathcal{G}}$ be the σ-algebra generated by $\mathcal{G}$, and let $\delta = \delta(\varepsilon)$ be from Lemma 7.16. We claim *that there exists* $E \in \Sigma_{\mathcal{G}}^+$ *such that for every* $j = 1, \dots, s$, M_j $(b + \tau/2)$-*norms every* δ-*elementary* $\mathcal{G}$-*sign* x *with* $\operatorname{supp} x \subseteq E$.

Assuming that the claim is false, by Zorn's lemma and the arguments used in the proof of Lemma 7.22, there exists a disjoint finite or countable sequence $(y_i)_{i \in I}$ of δ-elementary $\mathcal{G}$-signs such that

$$\sum_{i \in I} \|y_i\| = 1 \tag{7.36}$$

and for every $i \in I$, there exists $j \in \{1, \dots, s\}$ with

$$M_j(y_i) < \|y_i\|\Big(b + \frac{\tau}{2}\Big)\,. \tag{7.37}$$

For each $j = 1, \dots, s$, let I_j be the set of all $i \in I$ for which (7.37) is satisfied. Since $\sum_{j=1}^{s} I_j = I$, by (7.36) we have that

$$\sum_{j=1}^{s} \sum_{i \in I_j} \|y_i\| = 1\,.$$

Choose $j_0 \in \{1, \dots, s\}$ so that

$$\sum_{i \in I_{j_0}} \|y_i\| \geq \frac{1}{s}\,. \tag{7.38}$$

Set $y = \sum_{i \in I_{j_0}} y_i$. By Lemma 7.16, y is an ε-elementary $\mathcal{G}$-sign, and by (7.37) and (7.38),

$$M_{j_0}(y) < \|y\|\Big(b + \frac{\tau}{2}\Big) \quad \text{and} \quad \|y\| \geq \frac{1}{s}\,. \tag{7.39}$$

Choose an elementary $\mathcal{G}$-sign x with $\|\widetilde{x} - \widetilde{y}\| < \varepsilon$. Since, by Lemma 7.13(a), $\|x - y\| < 2\varepsilon\|y\|$ and by (7.39), we have

$$\|x\| \geq \|y\| - \|x - y\| > \|y\|(1 - 2\varepsilon) \geq \frac{1 - 2\varepsilon}{s}\,. \tag{7.40}$$

Since $\frac{1-2\varepsilon}{s} \geq \frac{1}{2s}$, (7.34) implies that

$$M_{j_0}(\widetilde{x}) > \eta_{j_0} - 4s\,\varepsilon \geq b + \tau - 4s\,\varepsilon\,. \tag{7.41}$$

By (7.39), $M_{j_0}(\widetilde{x}) < b + \tau/2 + \varepsilon$. Hence, $b + \tau - 4s\,\varepsilon < b + \tau/2 + \varepsilon$ which implies that $\tau < 2(4s+1)\,\varepsilon$, a contradiction which proves the claim.

Let E be a set satisfying the claim. By Lemma 7.21, we choose a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ with $\operatorname{supp} \mathcal{G}' = E$ such that every member of $\mathcal{G}'$ is a δ-elementary $\mathcal{G}$-sign. By Lemma 7.16, for every $j = 1, \ldots, s$, M_j $(b + \tau/2)$-norms every ε-elementary $\mathcal{G}'$-sign, and hence, for all j, M_j b^+-norms $\mathcal{G}'$. □

Lemma 7.27. *Let $b > 0$ and $n \in \mathbb{N}$. Suppose that all the sets $M_1, \ldots, M_k \subseteq B_{L_\infty}$ b^+-norm a tree of signs $\mathcal{G}$. Then there exists a tree of signs $\mathcal{G}'_0$ related to $\mathcal{G}$ with the following property:*

Given any n elementary $\mathcal{G}'_0$-signs $x_1, \ldots, x_n$ with $\operatorname{supp} x_i = \operatorname{supp} \mathcal{G}'_0$ *for all i, there exists a tree $\mathcal{G}_1$ related to $\mathcal{G}$ such that $M_j^{x_i > b}$ b^+-norms $\mathcal{G}'$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$.*

Proof. Let $\tau \in (0, 1)$ be the number from Lemma 7.25 and set

$$\varepsilon = \frac{\tau}{6(4nk+1)}\,. \tag{7.42}$$

By Lemma 7.25, choose a tree of signs $\mathcal{G}' = (g'_\alpha)_{\alpha \in \mathcal{D}}$ related to $\mathcal{G}$, and numbers $\eta_j \geq b + \tau$ for $j = 1, \ldots, k$ so that (7.32) is satisfied. Define $E_\alpha = \operatorname{supp} g'_\alpha$ for each $\alpha \in \mathcal{D}$. Let $\mathcal{G}'_0$ and $\mathcal{G}'_1$ be the pieces of $\mathcal{G}'$ consisting of all $g \in \mathcal{G}'$ with $\operatorname{supp} g \subseteq E_{(0)}$ and $\operatorname{supp} g \subseteq E_{(1)}$, respectively.

Let $x_1, \ldots, x_n$ be elementary $\mathcal{G}'_0$-signs with $\operatorname{supp} x_j = E_{(0)}$. We now fix $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$ and claim that

For every elementary $\mathcal{G}'_1$-sign y we have

(a) $M_j^{x_i > b}(\widetilde{y}) < \eta_j + \varepsilon$;

(b) $M_j^{x_i > b}(\widetilde{y}) > \eta_j - 3\varepsilon$ *provided* $\operatorname{supp} y = \operatorname{supp} \mathcal{G}'_1$.

Indeed, (a) immediately follows from (7.32):

$$M_j^{x_i > b}(\widetilde{y}) \leq M_j(\widetilde{y}) < \eta_j + \varepsilon\,.$$

To show (b), assume $\operatorname{supp} y = E_{(1)}$ and observe that by (7.32), there exists $m = m(i, j, y) \in M_j$ such that

$$\frac{1}{2}\Big|\int_{[0,1]} m(\widetilde{x_i} + \widetilde{y})\,\mathrm{d}\mu\Big| = \Big|\int_{[0,1]} m(\widetilde{x_i + y})\,\mathrm{d}\mu\Big| > \eta_j - \varepsilon\,. \tag{7.43}$$

By (7.32), we have that

$$\Big|\int_{[0,1]} m\widetilde{x_i}\,\mathrm{d}\mu\Big| < \eta_j + \varepsilon \ \text{ and } \ \Big|\int_{[0,1]} m\widetilde{y}\,\mathrm{d}\mu\Big| < \eta_j + \varepsilon\,. \tag{7.44}$$

Now we are going to show that

$$\Big|\int_{[0,1]} m\widetilde{y}\,\mathrm{d}\mu\Big| > \eta_j - 3\varepsilon \ \text{ and } \ \Big|\int_{[0,1]} m\widetilde{x_i}\,\mathrm{d}\mu\Big| > \eta_j - 3\varepsilon\,. \tag{7.45}$$

If the first inequality were not true, we would obtain by (7.44) that

$$\frac{1}{2}\Big|\int_{[0,1]} m(\widetilde{x_i}+\widetilde{y})\,\mathrm{d}\mu\Big| < \frac{\eta_j - 3\varepsilon}{2} + \frac{\eta_j+\varepsilon}{2} = \eta_j - \varepsilon\,,$$

which contradicts (7.43). The second inequality in (7.45) is proved analogously.

By (7.45) and by the choice of η_j, we have that

$$\Big|\int_{[0,1]} m\widetilde{x_i}\,\mathrm{d}\mu\Big| > \eta_j - 3\varepsilon \ge b + \tau - 3\varepsilon > b\,,$$

and hence, $m \in M_j^{x_i>b}$. Thus, by (7.45),

$$M_j^{x_i>b}(\widetilde{y}) \ge \Big|\int_{[0,1]} m\widetilde{y}\,\mathrm{d}\mu\Big| > \eta_j - 3\varepsilon\,,$$

proving (b).

Now letting $s = nk$, we have that the hypotheses of Lemma 7.26 are satisfied for the sets $M_j^{x_i>b}$ for $j = 1,\dots,k$, $i = 1,\dots,n$, for $\mathcal{G}_1'$ instead of $\mathcal{G}$ and 3ε instead of ε. Lemma 7.26 yields that there exists a tree of signs $\mathcal{G}_1$ related to $\mathcal{G}_1'$ such that $M_j^{x_i>b}$ b^+-norms $\mathcal{G}_1$ for all $i = 1,\dots,n$ and $j = 1,\dots,k$. It remains to observe that, since $\mathcal{G}_1'$ is a piece $\mathcal{G}'$ and $\mathcal{G}'$ is related to $\mathcal{G}$, by Lemma 7.17, $\mathcal{G}_1$ is related to $\mathcal{G}$. □

Lemma 7.28. *Let $0 < a < b$ be numbers. Suppose that every set $M_1,\dots,M_k \subseteq B_{L_\infty}$ b^+-norms a tree of signs $\mathcal{G}$. Let $n \ge 1 + \frac{k}{a^2}$, and $x_1,\dots,x_n$ be orthogonal (i.e. $\int_{[0,1]} x_i x_j\,\mathrm{d}\mu = 0$ for $i \ne j$) signs with common support. Then there exist $i \in \{1,\dots,n\}$ and a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ such that $M_j^{x_i<a}$ b^+-norms $\mathcal{G}'$ for every $j = 1,\dots,k$.*

Proof. Let $\mathcal{G} = (g_\alpha)_{\alpha\in\mathbb{D}}$. For convenience of the notation, without loss of generality we assume in addition that $\operatorname{supp} x_i = [0,1]$ for each $i = 1,\dots,n$. We first do some elementary counting. Assume $M \subseteq B_{L_\infty}$ and $X \subseteq X_0 = \{x_1,\dots,x_n\}$ with the number of elements $|X| \ge a^{-2}$. If $m \in M$ then, by Bessel's inequality,

$$\sum_{x\in X}\Big|\int_{[0,1]} mx\,\mathrm{d}\mu\Big|^2 \le \|m\|_{L_2}^2 \le 1\,. \tag{7.46}$$

Hence, $|\{x \in X : |\int_{[0,1]} mx \,\mathrm{d}\mu| \geq a\}| \leq 1/(a^2)$, and therefore, for each $m \in M$,

$$\left|\left\{x \in X : \left|\int_{[0,1]} mx \,\mathrm{d}\mu\right| < a\right\}\right| \geq |X| - \frac{1}{a^2}\,. \tag{7.47}$$

Given $Z \subseteq X_0$, we set

$$M_Z = \left\{m \in M : \left|\int_{[0,1]} mx \,\mathrm{d}\mu\right| < a \ \text{ for all } \ x \in Z\right\}.$$

Then by (7.47) we have

$$M = \bigcup\left\{M_Z : Z \subseteq X \quad \text{and} \quad |Z| \geq |X| - \frac{1}{a^2}\right\}. \tag{7.48}$$

By Lemma 7.23 we have the following:

(a) *If M b^+-norms a tree of signs $\mathcal{H}$ then there exists a tree of signs $\mathcal{H}'$ related to $\mathcal{H}$ and a subset $Z \subseteq X$ with $|Z| \geq |X| - \frac{1}{a^2}$ such that M_Z b^+-norms $\mathcal{H}'$.*

Recursively, we choose subsets $X_1, \ldots, X_k$ of X_0 and trees of signs $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_1, \ldots, \mathcal{G}_k$ so that for every $i = 1, \ldots, k$,

(b) $X_i \subseteq X_{i-1}$ *and* $|X_i| \geq n - \frac{i}{a^2}$.

(c) $\mathcal{G}_i$ *is related to* $\mathcal{G}_{i-1}$ *and* $(M_i)_{X_i}$ b^+*-norms* $\mathcal{G}_i$.

This is possible since, if $i \in \{1, \ldots, k-1\}$, and if X_{i-1} and $\mathcal{G}_{i-1}$ have been chosen, then, by (a), there exists $\mathcal{G}_i$ related to $\mathcal{G}_{i-1}$ such that $(M_i)_{X_i}$ b^+-norms $\mathcal{G}_i$ with

$$|X_i| \geq |X_{i-1}| - \frac{1}{a^2} \overset{\text{by (b)}}{\geq} n - \frac{i-1}{a^2} - \frac{1}{a^2} = n - \frac{i}{a^2}\,.$$

Thus, the recursive construction is completed.

We have $|X_k| \geq n - \frac{k}{a^2} \geq 1$. Suppose that $x \in X_k$. Then $x \in X_i$ for every $i \in \{1, \ldots, k\}$. Since $\|x\| = 1$, we obtain that $(M_i)_{X_i} \subseteq M_i^{x<a}$. Then by (c) we have $M_i^{x<a}$ b^+-norms $\mathcal{G}_i$. By Lemma 7.17, $\mathcal{G}_k$ is related to $\mathcal{G}_i$. By Lemma 7.18, there exists a piece $\mathcal{G}'$ of $\mathcal{G}_k$ so that $M_i^{x<a}$ b^+-norms $\mathcal{G}'$. Since $\mathcal{G}_k$ is related to $\mathcal{G}$, by Lemma 7.17, so too is $\mathcal{G}'$. □

Lemma 7.29. *Let $0 < a < b$ be numbers. Suppose that every set $M_1, \ldots, M_k \subseteq B_{L_\infty}$ b^+-norms a tree of signs $\mathcal{G}$. Then there exists a sign x and a tree of signs $\mathcal{G}'$ related to $\mathcal{G}$ such that for every $i = 1, \ldots, k$, $M_i^{x>b}$ and $M_i^{x<a}$ b^+-norm $\mathcal{G}'$.*

Proof. We fix $n \in \mathbb{N}$ with $n \geq 1 + \frac{k}{a^2}$ and choose a tree of signs $\mathcal{G}_0'$ satisfying the conclusion of Lemma 7.27. Then by Lemma 7.11 we choose orthogonal elementary $\mathcal{G}_0'$-signs $x_1, \ldots, x_n$ with $\operatorname{supp} x_i = \mathcal{G}_0'$ for all i. By Lemma 7.27, we choose a tree of signs $\mathcal{G}''$ related to $\mathcal{G}$ such that $M_j^{x_i>b}$ b^+-norms $\mathcal{G}''$, for every $i = 1, \ldots, n$

and $j = 1, \dots, k$. In particular, M_j b^+-norms $\mathcal{G}''$, for every $j = 1, \dots, k$. By Lemma 7.28 we choose $i \in \{1, \dots, n\}$ and a tree of signs $\mathcal{G}_1'$ related to $\mathcal{G}''$ such that $M_j^{x_i<a}$ b^+-norms $\mathcal{G}_1'$, for every $j = 1, \dots, k$. Finally, by Lemma 7.18, we choose a piece $\mathcal{G}'$ of $\mathcal{G}_1'$ such that $M_j^{x_i>b}$ b^+-norms $\mathcal{G}'$ for every $j = 1, \dots, k$. Since $\mathcal{G}'$ is a piece of $\mathcal{G}_1'$, $\mathcal{G}'$ is related to $\mathcal{G}_1'$. And since $\mathcal{G}_1'$ is related to $\mathcal{G}''$ and $\mathcal{G}''$ is related to $\mathcal{G}$, by Lemma 7.17, $\mathcal{G}'$ is related to $\mathcal{G}$. □

Proof of Theorem 7.4. Let $\mathcal{G}$ be the dyadic tree of signs, that is, the L_∞-normalized Haar system without the first element. Without loss of generality we assume that $\|T\| = 1$. Identifying L_1^* with L_∞, we set $M = T^* B_{X^*}$. By the theorem assumption, M δ-norms every elementary $\mathcal{G}$-sign x with $\operatorname{supp} x = [0, 1]$. Fix arbitrary numbers $0 < a < b < \delta$. By Lemma 7.22, choose a tree of signs $\mathcal{G}'$ such that M b^+-norms $\mathcal{G}'$.

Now we construct recursively a sequence $(x_n)_{n=1}^\infty$ of signs and a sequence $(\mathcal{G}_n)_{n=1}^\infty$ of trees of signs with the following property:

$$\text{for all } n \in \mathbb{N}, \varepsilon_j = \pm 1, \ j = 1, \dots, n, \text{ the set } \bigcap_{j=1}^{n} B_j^{\varepsilon_j} \ b^+\text{-norms } \mathcal{G}_n, \tag{7.49}$$

where

$$B_n^1 = \Big\{m \in M : \Big|\int_{[0,1]} m\widetilde{x}_n \, \mathrm{d}\mu\Big| > b\Big\}, B_n^{-1} = \Big\{m \in M : \Big|\int_{[0,1]} m\widetilde{x}_n \, \mathrm{d}\mu\Big| < a\Big\} .$$

We proceed as follows. By Lemma 7.29, for $k = 1$ and $M_1 = M$ we choose a tree of signs $\mathcal{G}_1$ and a sign x_1 such that $M^{x>b}$ and $M^{x<a}$ b^+-norm $\mathcal{G}_1$. Suppose that for a given $n \in \mathbb{N}$ signs $x_1, \dots, x_n$ and trees $\mathcal{G}_1, \dots, \mathcal{G}_n$ have been chosen to satisfy (7.49). Let $k = 2^n$ and $M_1, \dots, M^{2^n}$ be an enumeration of the sets $\bigcap_{j=1}^n B_j^{\varepsilon_j}$ over all choices of $\varepsilon_j = \pm 1$, $j = 1, \dots, n$. By Lemma 7.29, we choose a sign x_{n+1} and a tree of signs $\mathcal{G}_{n+1}$ related to $\mathcal{G}_n$ so that $M_i^{x_{n+1}>b}$ and $M_i^{x_{n+1}<a}$ b^+-norm $\mathcal{G}_{n+1}$ for every $i = 1, \dots, 2^n$. Since

$$\Big(\bigcap_{j=1}^{n} B_j^{\varepsilon_j}\Big)^{x_{n+1}>b} = \bigcap_{j=1}^{n+1} B_j^{\varepsilon_j} \quad \text{for any sign numbers } (\varepsilon_j)_{j=1}^n \text{ and } \varepsilon_{n+1} = 1, \text{ and}$$

$$\Big(\bigcap_{j=1}^{n} B_j^{\varepsilon_j}\Big)^{x_{n+1}<a} = \bigcap_{j=1}^{n+1} B_j^{\varepsilon_j} \quad \text{for any sign numbers } (\varepsilon_j)_{j=1}^n \text{ and } \varepsilon_{n+1} = -1,$$

we have that (7.49) is satisfied for $n + 1$, and the recursive construction is completed.

In particular, it follows that the sequence $(B_n^1, B_n^{-1})_{n=1}^\infty$ is Boolean independent. Then we use Lemma 7.20 for $S = M$, $f_n(s) = \int_{[0,1]} s\widetilde{x}_n \, \mathrm{d}\mu$ (in this case we have

$A_n = B_n^1$ and $B_n = B_n^{-1}$). Thus, given any $n \in \mathbb{N}$ and any scalars $(c_j)_{j=1}^n$, for $x = \sum_{j=1}^n c_j \widetilde{x}_j$, we have

$$\Big\| \sum_{j=1}^n c_j T x_j \Big\| = \|Tx\| = \sup_{f \in B_{X^*}} \big| f(Tx) \big| = \sup_{f \in B_{X^*}} \Big| \int_{[0,1]} (T^* f) \cdot x \, \mathrm{d}\mu \Big|$$

$$= \sup_{m \in M} \Big| \int_{[0,1]} m x \, \mathrm{d}\mu \Big| = \sup_{m \in M} \Big| \sum_{j=1}^n c_j f_j(m) \Big|$$

$$\overset{\text{by (7.23)}}{\geq} \frac{b-a}{2} \sum_{j=1}^n |c_j| \geq \frac{b-a}{2} \|x\| \geq \frac{b-a}{2\,\|T\|} \|Tx\| .$$

Thus, both sequences $(\widetilde{x}_i)$ and $(T\widetilde{x}_i)$ are equivalent to the unit vector basis of ℓ_1. □

7.2 Rosenthal's characterization of narrow operators on L_1 and some related results

In this section we prove the following remarkable result of Rosenthal [128].

Theorem 7.30. *An operator $T \in \mathcal{L}(L_1)$ is narrow if and only if for every $A \in \Sigma^+$ the restriction $T|_{L_1(A)}$ of T to $L_1(A)$ is not an isomorphic embedding.*

We present a proof of Theorem 7.30 in the context of the general theory. Proofs of auxiliary statements combine the original ideas of Enflo–Starbird [37], Kalton [66], Rosenthal [128], the results of two papers by O. Maslyuchenko, Mykhaylyuk and Popov [92, 93], and several private communications of Mykhaylyuk.

Every operator from L_1 to a Banach space X almost attains its norm at a positive cone of some set

We present here a helpful result that has been discovered and originally proved by different authors (in particular, by Shvidkoy in [131] and by Mykhaylyuk and Popov in [100]). However, the first author of it (to the best of our knowledge) is Rosenthal [128]. In our proof we follow [100].

Theorem 7.31. *Let X be a Banach space and $T \in \mathcal{L}(L_1, X)$. For every $\varepsilon > 0$ there exists an $A \in \Sigma^+$ such that $\|Tx\| \geq (\|T\| - \varepsilon)\|x\|$ for each $x \in L_1^+(A)$.*

First we prove the following statement.

Lemma 7.32. *Let $M \subseteq B(L_1)$. Assume that for each $A \in \Sigma^+$ and each $\varepsilon > 0$, there exists $x \in M$ such that $\int_A x \, \mathrm{d}\mu > 1 - \varepsilon$. Then the closed absolute convex hull $\widetilde{M}$ of M coincides with the unit ball B_{L_1}.*

Proof. Let D be the set of all $f \in L_\infty$ such that $\mu\{t : |f(t)| = \|f\|\} > 0$. First we prove that for any $f \in D$ we have

$$\sup_{x \in M} \int_{[0,1]} f x \, \mathrm{d}\mu = \|f\| . \tag{7.50}$$

Fix $\varepsilon > 0$. Choose $A \in \Sigma^+$ such that $|f(t)| = \|f\|$ for every $t \in A$, and, moreover, f has a common sign on A, say $f(t) = \|f\|$. Let $x \in M$ be so that $\int_A x \, \mathrm{d}\mu > 1 - \varepsilon$. Then we have

$$\begin{aligned}
\int_{[0,1]} f x \, \mathrm{d}\mu &\geq \int_A f x \, \mathrm{d}\mu - \int_{[0,1] \setminus A} |f||x| \, \mathrm{d}\mu \\
&\geq \|f\|(1-\varepsilon) - \|f\| \left(\|x\| - \int_A |x| \, \mathrm{d}\mu \right) \\
&= \|f\|(1 - \varepsilon - \|x\|) + \|f\| \int_A |x| \, \mathrm{d}\mu \\
&\geq -\varepsilon \|f\| + \|f\| \int_A x \, \mathrm{d}\mu \geq \|f\|(1 - 2\varepsilon) .
\end{aligned}$$

Thus, (7.50) is proved.

Now suppose that the lemma is not true and there exists $x_0 \in B_{L_1} \setminus \widetilde{M}$. By the Hahn–Banach Theorem, there are $f_0 \in L_\infty$ and $\delta > 0$ such that $f_0(x_0) = 1$ and $f_0(x) \leq 1 - \delta$ for each $x \in \widetilde{M}$. Since D is dense in L_∞, there exists $f \in D$ with $\|f - f_0\| < \frac{\delta}{3}$. Thus, for each $x \in \widetilde{M}$,

$$\begin{aligned}
\int_{[0,1]} f x \, \mathrm{d}\mu &= \int_{[0,1]} f_0 x \, \mathrm{d}\mu + \int_{[0,1]} (f - f_0) x \, \mathrm{d}\mu \\
&\leq 1 - \delta + \|f - f_0\| \|x\| < 1 - \frac{2\delta}{3} < \|f\| - \frac{\delta}{3} .
\end{aligned}$$

This contradicts (7.50). □

Proof of Theorem 7.31. Supposing the contrary, we obtain that there is $\delta \in (0, \|T\|)$ such that for each $A \in \Sigma^+$ there exists $x \in L_1^+(A)$, with $\|x\| = 1$ and $\|Tx\| < \|T\| - \delta$. Let

$$M = \big\{ x \in B(L_1) : \|Tx\| \leq \|T\| - \delta \big\} .$$

On the one hand, the closed absolute convex hull of M equals M, that is, $\widetilde{M} = M$. On the other hand, M satisfies the assumptions of Lemma 7.32, and hence, $M = B(L_1)$, a contradiction. □

Disjointness-preserving operators

Recall that two elements x and y of a Köthe–Banach space E on a measure space (Ω, Σ, μ) are called *disjoint* if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$. In this case we write $x \perp y$. An element $x \in E$ is said to be a *fragment* of $y \in E$ provided $x \perp (y - x)$. An operator $T \in \mathcal{L}(E_1, E_2)$ between Köthe–Banach spaces E_i on measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, respectively, is called *disjointness preserving* ("*atom*," in the terminology of Kalton and Rosenthal, which is less convenient) if for each $x, y \in E_1$ the condition $x \perp y$ implies $Tx \perp Ty$. We use the abbreviation *d.p.o.* for these operators. Obviously, a d.p.o. T maps fragments of an arbitrary element $y \in L_1$ to fragments of Ty (cf. Section 1.6).

Using the definition of a d.p.o., we obtain the following consequence of Theorem 7.31.

Corollary 7.33. *Let $T \in \mathcal{L}(L_1, L_1(\mu))$ be a d.p.o. Then for every $\varepsilon > 0$ there exists $A \in \Sigma^+$ such that $\|Tx\| \geq (\|T\| - \varepsilon)\|x\|$ for each $x \in L_1(A)$. In particular, if $T \neq 0$ then there is $A \in \Sigma^+$ such that the restriction $T|_{L_1(A)}$ is an into isomorphism.*

The proof easily follows from the observation that $\|Tx\| = \|Tx^+ - Tx^-\| = \|Tx^+\| + \|Tx^-\|$ for each $x \in L_1$, because T is a d.p.o.

We need the following well-known characterization of d.p.o. on L_1, due to Abramovich [1], who proved it for general vector lattices (cf. Rosenthal [128] for the case of L_1). Before stating it, we introduce some terminology.

We say that a measurable function $\sigma : [0,1] \to [0,1]$ is *almost injective* if for every $B \in \Sigma$ with $\mu(B) = 0$ we have $\mu(\sigma^{-1}(B)) = 0$. For measurable functions f, g the notation $f \sim g$ means that they are equivalent, i.e. $\mu\{t : f(t) \neq g(t)\} = 0$.

Proposition 7.34.

(a) *A measurable function $\sigma : [0,1] \to [0,1]$ is almost injective if and only if*

$$\text{for every } x \in L_0,\ x \circ \sigma \text{ is well defined as an element of } L_0\,, \tag{7.51}$$

that is, if $f, g : [0,1] \to [0,1]$, are measurable and $f \sim g$ then $f \circ \sigma \sim g \circ \sigma$.

(b) *If σ is almost injective then $\sigma^{-1}(B) \in \Sigma$, for every $B \in \Sigma$.*

Proof. (a) Let $B = \{t : f(\sigma(t)) \neq g(\sigma(t))\}$ and $A = \{\tau : f(\tau) \neq g(\tau)\}$. Thus, (7.51) holds if and only if $\mu(B) = 0$ whenever $\mu(A) = 0$. It remains to notice that $B = \sigma^{-1}(A)$.

(b) Let $B \in \Sigma$. Since for every $n \in \mathbb{N}$, there is a closed set $F_n \subseteq B$ with $\mu(B \setminus F_n) < 1/n$, we have $B = B' \sqcup B''$ where $B' = \bigcup_{n=1}^{\infty} F_n$ is an F_σ-set and $B'' = B \setminus B'$, $\mu(B'') = 0$. Then $\sigma^{-1}(B') \cup \sigma^{-1}(B'')$ where $\sigma^{-1}(B'), \sigma^{-1}(B'') \in \Sigma$. □

Proposition 7.35. *An operator $T \in \mathcal{L}(L_1)$ is a d.p.o. if and only if there exist measurable functions $f : [0,1] \to \mathbb{R}$ and $\sigma : [0,1] \to [0,1]$ so that σ is almost injective on* supp f,

$$(Tx)(t) = f(t) \cdot x(\sigma(t)) \ \text{ a.e. for each } \ x \in L_1 \tag{7.52}$$

and

$$K = \sup\Big\{\frac{1}{\mu(B)}\int_{\sigma^{-1}(B)} |f|\,\mathrm{d}\mu : \mu(B) > 0\Big\} < \infty\,.$$

Moreover, $\|T\| = K$.

Proof. The "if" part is obvious. To prove "only if," let f be a representative of the equivalence class $T\mathbf{1}_{[0,1]}$. For $n \in \mathbb{N}$ and $k = 1,\dots,2^n$ denoted by I_n^k there is the dyadic interval $[2^{-n}\cdot(k-1), 2^{-n}\cdot k)$. Let $f_0^1 = f$. Choose a representative f_n^k of the equivalence class $T\mathbf{1}_{I_n^k}$ in such a way that for $J_n^k = \operatorname{supp} f_n^k$, $n \in \mathbb{N}$ and $k = 1,\dots,2^n$, we have $J_n^k = J_{n+1}^{2k-1} \sqcup J_{n+1}^{2k}$. Then for all $\tau \in J_0^1$ there exists a unique sequence of nested sets $(J_n^{k_n})_0^\infty$ containing τ. Thus, we can define $\sigma(\tau) = t$, where $\{t\} = \bigcap_{n=0}^\infty \overline{I_n^{k_n}}$ (here $\overline{I_n^{k_n}}$ denotes the closure of the interval $I_n^{k_n}$). Moreover, for all $\tau \in [0,1]\setminus J_0^1$ we set $\sigma(\tau) = 1$. It is a standard technical exercise to show that the desired measurable functions are well defined.

Show, for example, that σ is almost injective. Let $B \in \Sigma$ and $\mu(B) = 0$. Assume first that $|f(t)| \ge \delta$ for all $t \in J_0^1$ and some $\delta > 0$. Then

$$\mu(J_n^k) = \|\mathbf{1}_{J_n^k}\| \le \delta^{-1}\|f\cdot\mathbf{1}_{J_n^k}\| = \delta^{-1}\|T\mathbf{1}_{I_n^k}\| \le \delta^{-1}\|T\|\mu(I_n^k)\,. \tag{7.53}$$

Let $B \in \Sigma$, $\mu(B) = 0$, $\varepsilon > 0$, and $(I_{n_j}^{k_j})_{j=1}^\infty$ be a sequence with $B \subseteq \bigcup_{j=1}^\infty I_{n_j}^{k_j}$ and $\sum_{j=1}^\infty \mu(I_{n_j}^{k_j}) < \varepsilon\delta\|T\|^{-1}$. Then $\sigma^{-1}(B) \subseteq \bigcup_{j=1}^\infty \sigma^{-1}(I_{n_j}^{k_j}) = \bigcup_{j=1}^\infty J_{n_j}^{k_j}$, and by (7.53),

$$\sum_{j=1}^\infty \mu(J_{n_j}^{k_j}) \le \delta^{-1}\|T\|\sum_{j=1}^\infty \mu(I_{n_j}^{k_j}) < \varepsilon\,.$$

This proves that $\mu(\sigma^{-1}(B)) = 0$. To prove almost injectivity for the general case, we split $J_0^1 = \bigsqcup_{n=0}^\infty C_n$, where $C_0 = \{|f| \ge 1\}$ and $C_n = \{(n+1)^{-1} \le |f| < n^{-1}\}$. By the above, σ is almost injective on each C_n, and so it is almost injective on $\operatorname{supp} f$. □

Proposition 7.36. *Let* $0 \ne T \in \mathcal{L}(L_1)$ *be a d.p.o. Then there are* $A, B \in \Sigma^+$ *and a sub-*σ*-algebra* Σ_1 *of* $\Sigma(B)$ *such that* $T|_{L_1(A)} : L_1(A) \to L_1(B,\Sigma_1)$ *is an isomorphism.*

Proof. Let f and σ be as in Proposition 7.35. By Corollary 7.33, there exists $A_0 \in \Sigma^+$ so that $T_0 = T|_{L_1(A_0)} : L_1(A_0) \to L_1$ is an into isomorphism. Let $B_0 = \operatorname{supp} T\mathbf{1}_{A_0}$. Observe that if $x \in L_1(A_0)$ then $x \circ \sigma \in L_1(B_0)$ (to prove this, first consider x of the form $x = \mathbf{1}_A$, $A \in \Sigma(A_0)$, then prove it for simple functions, and finally use the density argument) and hence, $Tx \in L_1(B_0)$. Let $a > 0$ so that $\mu(A) > 0$, where $A = \{t \in A_0 : |f(t)| \ge a\}$. By Proposition 7.34(b), $B = \sigma^{-1}(A) \in \Sigma$. If $x \in L_1(A)$ then $x \circ \sigma \in L_1(B)$ and hence $T_1 = T|_{L_1(A)} : L_1(A) \to L_1(B)$. Since

$A \subseteq A_0$, T_1 is an into isomorphism. Let $\Sigma_1 = \{C \in \Sigma(B) : \sigma^{-1}(C) \in \Sigma(A)\}$. By almost injectivity of σ, Σ_1 is a sub-σ-algebra of $\Sigma(B)$. For surjectivity of $T_1 : L_1(A) \to L_1(B, \Sigma_1)$, by (7.52), it is enough to notice that for each $y \in L_1(B)$, it follows from the definition of B and Σ_1 that

$$x(\cdot) = \frac{y(\sigma^{-1}(\cdot))}{f(\sigma^{-1}(\cdot))} \in L_1(A) .$$

□

Pointwise absolutely convergent series of operators

Definition 7.37. Let X, Y be Banach spaces. A series of operators $\sum_{n=1}^{\infty} T_n$, $T_n \in \mathcal{L}(X, Y)$ is said to be *pointwise absolutely convergent* if for every $x \in X$ the series $\sum_{n=1}^{\infty} \|T_n x\|$ converges.

Notice that if a series $\sum_{n=1}^{\infty} T_n$ is pointwise absolutely convergent then it is pointwise convergent to some operator $T \in \mathcal{L}(X, Y)$, and there exists a constant $K \in [0, +\infty)$ such that

$$\sum_{n=1}^{\infty} \|T_n x\| \le K\|x\| . \tag{7.54}$$

Indeed, the first fact follows immediately from the Uniform Boundedness Principle. To see that the second assertion is true, we consider the sequence of operators $S_n : X \to \ell_1(Y)$, defined by $Sx = (T_1 x, \dots, T_n x, 0, \dots)$ for each $n \in \mathbb{N}$ and $x \in X$. Since the series $\sum_{n=1}^{\infty} T_n$ is pointwise absolutely convergent, the sequence (S_n) is pointwise bounded and hence, is uniformly bounded, that gives (7.54).

The notion of pointwise absolute convergence was considered by Rosenthal in [128] under the name of “strong ℓ_1-convergence,” and recently by L. Kadets and V. Kadets in [51], from where we took its current name. Later in Chapter 10 we will show that for operators on L_1 this notion coincides with the notion of order convergence of operators on the vector lattice $\mathcal{L}(L_1)$.

First characterization of pseudo-embeddings on L_1

Recall that an operator $T \in \mathcal{L}(L_1)$ is a pseudo-embedding if there exists an absolutely order summable family $(T_j)_{j \in J}$ of d.p.o. in $\mathcal{L}(L_1)$ such that $T = \sum_{j \in J} T_j$ (cf. Definition 1.32). Theorem 1.33 implies that the set $L_{pe}(L_1)$ of all pseudo-embeddings on L_1 is the band generated by the d.p.o. on L_1.

Our first characterization of pseudo-embeddings on L_1-spaces is the following.

Theorem 7.38. *An operator $T \in \mathcal{L}(L_1(\mu), L_1(\nu))$ is a pseudo-embedding if and only if T equals the sum $T = \sum_{N=1}^{\infty} T_n$ of a pointwise absolutely converging series of d.p.o. $T_n \in \mathcal{L}(L_1(\mu), L_1(\nu))$.*

Proof. By Definition 1.19, a series $\sum_{n\in\mathbb{N}} T_n$ is absolutely order convergent if and only if the series $S = \sum_{n\in\mathbb{N}} |T_n|$ is order convergent. Hence, $S = \sup_{n\in\mathbb{N}} \sum_{k=1}^n |T_k|$ and $Sx = \sum_{n=1}^\infty |T_n|\, x$ for each $x \in L_1^+$. Therefore,

$$\sum_{k=1}^n \|T_k x\| \le \sum_{k=1}^n \big\| \, |T_k|\, x \big\| = \Big\| \sum_{k=1}^n |T_k|\, x \Big\| \le \|Sx\| \le \|S\| \cdot \|x\|$$

for all $n \in \mathbb{N}$ and $x \in L_1^+$. Thus, (7.54) is satisfied with $K = \|S\|$.

Before we prove the converse, note that by Lemma 1.23 and the order continuity of L_1, for any $U \in \mathcal{L}(L_1)$ we have for all $x \in L_1^+$,

$$\big\| \, |U|\, x \big\| = \sup\Big\{ \Big\| \sum_{i=1}^m |U x_i| \Big\| : \, x = \sum_{i=1}^m x_i, \;\; x_i \in X^+, \;\; m \in \mathbb{N} \Big\} . \tag{7.55}$$

Suppose that $\sum_{n=1}^\infty \|T_n x\| \le K\|x\|$ for all $x \in L_1^+$ and some $K < \infty$. We will show that

$$\sum_{n=1}^\infty \big\| \, |T_n|\, x \big\| \le K\|x\| \quad \text{for all } \; x \in L_1^+ \, . \tag{7.56}$$

Fix $x \in L_1^+$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Using (7.55) for $U = T_k$, $k = 1, \ldots, n$, we choose $x_1, \ldots, x_m \in L_1^+$, so that $x = \sum_{i=1}^m x_i$ and

$$\Big\| \sum_{i=1}^m |T_k x_i| \Big\| \ge \big\| \, |T_k| x \, \big\| - \frac{\varepsilon}{n}$$

for $k = 1, \ldots, n$. Then

$$\begin{aligned}\sum_{k=1}^n \big\| \, |T_k| x \, \big\| &\le \sum_{k=1}^n \Big\| \sum_{i=1}^m |T_k x_i| \Big\| + \varepsilon \le \sum_{i=1}^m \sum_{k=1}^n \|T_k x_i\| + \varepsilon \\ &\le \sum_{i=1}^m K\|x_i\| + \varepsilon = K\|x\| + \varepsilon \, .\end{aligned}$$

Taking $\varepsilon \to 0$ and then $n \to \infty$, we obtain (7.56).

It is not hard to see that there exists $S \in \mathcal{L}(L_1)$ which extends by linearity the equality $Sx = \lim_n \sum_{k=1}^n |T_k|\, x$ for all $x \in L_1^+$ and such that $\|S\| \le K$ and $\sum_{k=1}^n |T_k| \le S$. Then, $\sup_{n\in\mathbb{N}} \sum_{k=1}^n |T_k|$ exists and therefore the series $\sum_{n\in\mathbb{N}} T_n$ is absolutely order convergent.

We will now show that the series are convergent, then the order sum $T' = \sum_{n\in\mathbb{N}} T_n$ equals the pointwise absolute sum $T'' = \sum_{n=1}^\infty T_n$.

It is enough to prove that $T'x = \sum_{n=1}^\infty T_n x$ holds for all $x \in L_1^+$.

Let $x \in L_1^+$. Then

$$\Big\|\Big(T' - \sum_{k=1}^{n} T_k\Big)x\Big\| = \Big\|\Big(\sum_{k>n} T_k\Big)x\Big\| \le \Big\|\Big(\sum_{k>n} |T_k|\Big)x\Big\|$$
$$= \Big\|\sum_{k=n+1}^{\infty} \big(|T_k|\,x\big)\Big\| \le \sum_{k=n+1}^{\infty} \big\|\,|T_k|\,x\big\| \le \|x\| \sum_{k=n+1}^{\infty} \big\|\,|T_k|\,\big\|.$$

Thus, letting $n \to \infty$, we obtain $T' = T''$. □

The main property of pseudo-embeddings on L_1

The name of a pseudo-embedding becomes clear from the next theorem saying that a pseudo-embedding is an "almost" isometric embedding when restricted to a suitable subspace $L_1(A)$.

Theorem 7.39. *Let $T \in \mathcal{L}(L_1)$ be a nonzero pseudo-embedding. Then for each $\varepsilon > 0$ there exists $A \in \Sigma^+$ such that the restriction $T|_{L_1(A)}$ is an into isomorphism with*

(a) $\|T|_{L_1(A)}\| \ge \|T\| - \varepsilon$;

(b) $\|T|_{L_1(A)}\| \cdot \|T|_{L_1(A)}^{-1}\| < 1 + \varepsilon$;

(c) *there exists a d.p.o. $U : L_1(A) \to L_1$ so that $\|T|_{L_1(A)} - U\| < \varepsilon$.*

Without the last part, Theorem 7.39 is due to the work by Rosenthal [128]. Claim (c) is new, and it will serve the proof of Theorem 7.80 below, which is also due to Rosenthal. More precisely, the claim on the existence of U plays the role of Alspach's result from [7], the proof of which, in turn, involves further results of Dor and Schechtman.

For the proof we need some lemmas.

Lemma 7.40. *Let $T_1, \dots, T_n \in \mathcal{L}(L_1)$ be a d.p.o. and $\varepsilon > 0$. Then there exist $\delta > 0$ and $B \in \Sigma$ such that $\mu(B) > 1 - \varepsilon$ and for any $A \in \Sigma$ with* diam $A < \delta$*, the operator $T : L_1(A) \to L_1(B)$ defined by*

$$Tx = \Big(\sum_{i=1}^{n} T_i x\Big) \cdot \mathbf{1}_B, \quad x \in L_1(A)\,,$$

is a d.p.o.

Proof. For $i = 1, \dots, n$, choose, by Proposition 7.35, measurable functions $x_i, \sigma_i : [0,1] \to [0,1]$ such that $T_i\,x(t) = x_i(t) \cdot x(\sigma_i(t))$, for almost all $t \in [0,1]$. For any $1 \le i < j \le n$ and $k \in \mathbb{N}$, let $A_{i,j}^{(k)} = \{t : 0 < |\sigma_i(t) - \sigma_j(t)| \le$

$k^{-1}\}$. Since $\lim_{k\to\infty}\mu(A_{i,j}^{(k)}) = 0$, there exists k_0 so that $\mu(C) < \varepsilon$, where $C = \bigcup_{1\le i<j\le n} A_{i,j}^{(k_0)}$. Let $B = [0,1]\setminus C$, $\delta = k_0^{-1}$, $A \in \Sigma$ be any set with diam $A < \delta$, and $T : L_1(A) \to L_1(B)$ be an operator defined in the lemma conditions. We are going to prove that T is a d.p.o., i.e. that for any disjoint measurable subsets $U, V \subseteq A$ we have that $(T\mathbf{1}_U)(t)\cdot(T\mathbf{1}_V)(t) = 0$ a.e. on B. Note that

$$\big(T\mathbf{1}_U\big)(t)\cdot\big(T\mathbf{1}_V\big)(t) \overset{\text{a.e.}}{=} \Big(\sum_{i=1}^n x_i(t)\mathbf{1}_U\big(\sigma_i(t)\big)\Big)\cdot\Big(\sum_{i=1}^n x_i(t)\mathbf{1}_V\big(\sigma_i(t)\big)\Big)\,.$$

We will show that

$$\Big(\sum_{i=1}^n x_i(t)\mathbf{1}_U\big(\sigma_i(t)\big)\Big)\cdot\Big(\sum_{i=1}^n x_i(t)\mathbf{1}_V\big(\sigma_i(t)\big)\Big) = 0\,,$$

for all $t \in B$. Supposing the contrary, there would exist $t \in B$ and indices $i, j \in \{1,\dots,n\}$ such that $\mathbf{1}_U(\sigma_i(t)) \neq 0$ and $\mathbf{1}_V(\sigma_j(t)) \neq 0$, i.e. $\sigma_i(t) \in U$ and $\sigma_j(t) \in V$. Since $U \sqcup V \subseteq A$ and diam $A < \delta$, we have $0 < |\sigma_i(t) - \sigma_j(t)| < \delta$. Thus, $t \in A_{i,j}^{(k_0)}$ or $t \in A_{j,i}^{(k_0)}$, that contradicts the choice of B. □

Lemma 7.41. *Let $A, B \in \Sigma^+$ and $T : L_1(A) \to L_1(B)$ be a pseudo-embedding with $T = \sum_{n=1}^\infty T_n$ where $T_n : L_1(A) \to L_1(B)$ is a d.p.o. for each $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \|T_n x\| \le K\|x\|$ for all $x \in L_1$. Then for each $\varepsilon > 0$ there exist $A_0 \in \Sigma(A)^+$ and $n \in \mathbb{N}$ such that*

$$\Big\|\Big(\sum_{k>n} T_k\Big)\Big|_{L_1(A_0)}\Big\| < \varepsilon\,.$$

Proof. Supposing the contrary, we obtain that for some $\varepsilon > 0$ and all $A' \in \Sigma(A)^+$ and $n \in \mathbb{N}$ we have

$$\Big\|\Big(\sum_{k>n} T_k\Big)\Big|_{L_1(A')}\Big\| \ge \varepsilon\,. \tag{7.57}$$

In particular, $\|T\| \ge \varepsilon$. By Theorem 7.31, there exists $A_1' \in \Sigma(A)^+$ so that $\|T\mathbf{1}_{A_1'}\| > \frac{\varepsilon}{2}\,\mu(A_1')$. Since $\lim_{n\to\infty}\|\sum_{i=1}^n T_i\mathbf{1}_{A_1'}\| = \|T\mathbf{1}_{A_1'}\|$, there is $n_1 \in \mathbb{N}$ such that $\|\sum_{i=1}^{n_1} T_i\mathbf{1}_{A_1'}\| > \frac{\varepsilon}{2}\,\mu(A_1')$. Hence,

$$\Big\|\Big(\sum_{i=1}^{n_1} T_i\Big)\big|_{L_1(A_1')}\Big\| > \frac{\varepsilon}{2}\,.$$

By Theorem 7.31, there exists $A_1 \in \Sigma(A_1')^+$ such that for any $A' \in \Sigma(A_1)^+$ we have

$$\Big\|\sum_{i=1}^{n_1} T_i\mathbf{1}_{A'}\Big\| \ge \frac{\varepsilon}{2}\,\mu(A')\,.$$

Let

$$S_1 = \Big(\sum_{i=1}^{n_1} T_i\Big)\Big|_{L_1(A_1)} .$$

By (7.57), $\|S_1\| \geq \varepsilon$. By Theorem 7.31, we choose $A'_2 \in \Sigma(A_1)^+$ so that $\|S_1 \mathbf{1}_{A'_2}\| > \frac{\varepsilon}{2}\,\mu(A'_2)$. Then there exists n_2 such that

$$\Big\|\sum_{i=n_1+1}^{n_2} T_i \mathbf{1}_{A'_2}\Big\| > \frac{\varepsilon}{2}\,\mu(A'_2) .$$

In particular, $\|(\sum_{i=n_1+1}^{n_2} T_i)|_{L_1(A'_2)}\| > \frac{\varepsilon}{2}$. Now we choose $A_2 \in \Sigma(A'_2)^+$ so that

$$\Big\|\sum_{i=n_1+1}^{n_2} T_i \mathbf{1}_{A'}\Big\| \geq \frac{\varepsilon}{2}\,\mu(A')$$

for every $A' \in \Sigma(A_2)^+$. Continuing this procedure, we obtain a decreasing sequence (A_n), $A_n \in \Sigma(A)^+$ and an increasing sequence of numbers (n_k) such that

$$\Big\|\sum_{i=n_{k-1}+1}^{n_k} T_i \mathbf{1}_{A'}\Big\| \geq \frac{\varepsilon}{2}\,\mu(A') ,$$

for each $k \in \mathbb{N}$ and each $A' \in \Sigma(A_k)^+$. Choose $k \in \mathbb{N}$ so that $k \cdot \frac{\varepsilon}{2} > K$. Then

$$K \cdot \mu(A_k) \geq \sum_{i=1}^{n_k} \|T_i \mathbf{1}_{A_k}\| \geq \sum_{j=1}^{k} \Big\|\sum_{i=n_{k-1}+1}^{n_k} T_i \mathbf{1}_{A_k}\Big\| \geq k \cdot \frac{\varepsilon}{2} \cdot \mu(A_k) ,$$

which is impossible. □

Proof of Theorem 7.39. Let $\varepsilon > 0$ and let $0 \neq T \in \mathcal{L}(L_1)$ be a pseudo-embedding. Let $T = \sum_{n=1}^{\infty} T_n$ be an absolutely order summable series of d.p.o. $T_n \in \mathcal{L}(L_1)$. Choose $\delta > 0$ so that

$$(1-\delta)^3(1-2\delta) - \delta > \max\Big\{1 - \frac{\varepsilon}{1+\varepsilon},\ 1 - \frac{\varepsilon}{\|T\|}\Big\} \text{ and } \delta\|T\| < \varepsilon . \tag{7.58}$$

By Theorem 7.31, there exists $A_1 \in \Sigma^+$ so that $\|Tx\| \geq (1-\delta)\|T\|\|x\|$ for each $x \in L_1^+(A_1)$. Obviously, $T|_{L_1(A_1)} = \sum_{n=1}^{\infty} T_n|_{L_1(A_1)}$ is an absolutely order summable series of d.p.o. from $L_1(A_1)$ to L_1 with $\|T|_{L_1(A)}\|(1-\delta)\|T\|$ for every $A \in \Sigma^+(A_1)$.

By Lemma 7.41, we choose $A_2 \in \Sigma(A_1)^+$ and an $n \in \mathbb{N}$ such that

$$\Big\|\Big(\sum_{i>n} T_i\Big)\Big|_{L_1(A_2)}\Big\| = \Big\|\Big(\sum_{i>n} T_i|_{L_1(A_1)}\Big)\Big|_{L_1(A_2)}\Big\| < \delta\|T\| . \tag{7.59}$$

Set $S_1 = (\sum_{i=1}^n T_i)|_{L_1(A_2)}$. Since $\|T|_{L_1(A_2)}\| \geq (1-\delta)\,\|T\|$, we obtain that

$$\begin{aligned}\|S_1\| &\geq \big\|T|_{L_1(A_2)}\big\| - \Big\|\Big(\sum_{i>n} T_i\Big)\Big|_{L_1(A_2)}\Big\| \\ &\geq (1-\delta)\,\|T\| - \delta\,\|T\| = (1-2\delta)\,\|T\|\,. \end{aligned} \tag{7.60}$$

By Theorem 7.31, there exists $A_3 \in \Sigma(A_2)^+$ so that $\|S_1 x\| \geq (1-\delta)\cdot\|S_1\|\cdot\|x\|$ for each $x \in L_1^+(A_3)$. Let $\delta_1 > 0$ be so that for any $B \in \Sigma$ with $\mu(B) > 1-\delta_1$,

$$\int_B |S_1 \mathbf{1}_{A_3}|\,\mathrm{d}\mu \geq (1-\delta)\int_{[0,1]} |S_1 \mathbf{1}_{A_3}|\,\mathrm{d}\mu\,.$$

By Lemma 7.40, choose $\gamma > 0$ and $B' \in \Sigma$ so that $\mu(B') > 1-\delta_1$, and for any set $A' \in \Sigma(A_3)^+$, the operator $S_{A'} : L_1(A') \to L_1(B')$ defined by

$$S_{A'}x = (S_1 x)\cdot \mathbf{1}_{B'} \;\text{ for each } x \in L_1(A') \tag{7.61}$$

is a d.p.o. whenever diam $A' < \gamma$.

Now we claim that there exists $A_4 \in \Sigma(A_3)^+$ with diam $A_4 < \gamma$ such that

$$\big\|(S_1 \mathbf{1}_{A_4})\cdot \mathbf{1}_{B'}\big\| \geq (1-\delta)^2\cdot\|S_1\|\cdot\mu(A_4)\,. \tag{7.62}$$

Suppose on the contrary that this is not true. Decompose $A_3 = \bigsqcup_{i=1}^m C_i$ into subsets with diam $C_i < \gamma$, $i = 1,\dots,m$. By the assumption, for each $i = 1,\dots,m$,

$$\big\|(S_1 \mathbf{1}_{C_i})\cdot \mathbf{1}_{B'}\big\| < (1-\delta)^2\cdot\|S_1\|\cdot\mu(C_i)\,.$$

Then

$$\begin{aligned}(1-\delta)^2\cdot\|S_1\|\cdot\mu(A_3) &= (1-\delta)^2\cdot\|S_1\|\cdot\sum_{i=1}^m \mu(C_i) > \sum_{i=1}^m \big\|(S_1\mathbf{1}_{C_i})\cdot\mathbf{1}_{B'}\big\| \\ &\geq \big\|(S_1\mathbf{1}_{A_3})\cdot\mathbf{1}_{B'}\big\| \geq (1-\delta)\cdot\|S_1\mathbf{1}_{A_3}\| \\ &\geq (1-\delta)^2\cdot\|S_1\|\cdot\mu(A_3)\,,\end{aligned}$$

which again is impossible. Thus, the claim is proved.

Let $A_4 \in \Sigma(A_3)^+$ satisfy (7.62). By (7.62) and the choice of γ and B', the operator S_{A_4} defined by (7.61) is a d.p.o. with

$$\|S_{A_4}\| \geq (1-\delta)^2\cdot\|S_1\|\,. \tag{7.63}$$

By Corollary 7.33, there exists $A_5 \in \Sigma(A_4)^+$ such that, for all $x \in L_1(A_5)$,

$$\|S_{A_4}x\| \geq (1-\delta)\cdot\|S_{A_4}\|\cdot\|x\|\,. \tag{7.64}$$

Since S_{A_4} is a d.p.o., so is $U = S_{A_4}|_{L_1(A)} : L_1(A) \to L_1$. Since $A \subseteq A'$ and $B \subseteq B'$, by (7.61), $U = S_{A_4}|_{L_1(A)} = S_1|_{L_1(A)}$. Thus, for any $x \in L_1(A)$, we have

$$\begin{aligned}
\|Tx\| &\ge \|S_1 x\| - \Big\|\sum_{i>n} T_i x\Big\| \overset{\text{by (7.59)}}{\ge} \|S_{A_4}x\| - \delta\|T\|\|x\| \\
&\ge (1-\delta)\|S_{A_4}\|\|x\| - \delta\|T\|\|x\| \\
&\overset{\text{by (7.63)}}{\ge} \Big((1-\delta)^3\|S_1\| - \delta\|T\|\Big)\|x\| \\
&\overset{\text{by (7.60)}}{\ge} \Big((1-\delta)^3(1-2\delta) - \delta\Big)\|T\|\|x\| \\
&\overset{\text{by (7.58)}}{>} \Big(1 - \frac{\varepsilon}{1+\varepsilon}\Big)\|T\|\|x\| = \frac{\|T\|}{1+\varepsilon}\|x\| \ge \frac{\big\|T\big|_{L_1(A)}\big\|}{1+\varepsilon}\|x\| \,.
\end{aligned} \tag{7.65}$$

Thus, $T|_{L_1(A)}$ is an into isomorphism with $\|T|_{L_1(A)}\|\|T|^{-1}_{L_1(A)}\| \le 1 + \varepsilon$. Moreover, (7.65) and (7.58) give

$$\|T\big|_{L_1(A)}\| \ge \Big(1 - \frac{\varepsilon}{\|T\|}\Big)\|T\| = \|T\| - \varepsilon \,.$$

It remains to observe that

$$\big\|T\big|_{L_1(A)} - U\big\| = \Big\|\Big(\sum_{i>n} T_i\Big)\Big|_{L_1(A)}\Big\| \overset{\text{by (7.59)}}{<} \delta\|T\| \overset{\text{by (7.58)}}{<} \varepsilon \,.$$

□

The Enflo–Starbird maximal function and λ-narrow operators

For an operator $T \in \mathcal{L}(L_1)$, we denote by $\lambda_T(t)$ the Enflo–Starbird maximal function

$$\lambda_T(t) = \lim_{n\to\infty} \max_{1\le k\le 2^n} \big|T\mathbf{1}_{I_n^k}\big|(t) \,,$$

for $t \in [0,1]$, where $I_n^k = [\frac{k-1}{2^n}, \frac{k}{2^n})$ and the limit is taken in the sense of L_1-norm; see Proposition 7.42 below for the proof that $\lambda_T(t)$ is well defined for almost all $t \in [0,1]$.

Formally the Enflo–Starbird function was introduced by Rosenthal in [128], however implicitly it appeared in [37] (1979). Its properties were used to study Enflo operators in $\mathcal{L}(L_1)$. Almost simultaneously similar ideas were used by Kalton in [66]. A generalized version of the Enflo–Starbird function for vector lattices will be used later in Section 10.4 to prove the main result of Chapter 10.

Proposition 7.42 (Enflo and Starbird [37]). *For every $T \in \mathcal{L}(L_1)$ the function $\lambda_T(t)$ is well defined for almost all $t \in [0,1]$, and the limit exists in the sense of L_1-norm.*

Proof. Let $T \in \mathcal{L}(L_1)$. For each $n \in \mathbb{N}$, we define $g_n \in L_1$ for $t \in [0,1]$, by

$$g_n(t) = \max_{1 \le k \le 2^n} \left| T\mathbf{1}_{I_n^k} \right|(t),$$

and an L_1-valued function v_n, defined on the finite algebra $\mathcal{F}_n$ generated by the dyadic intervals $(I_n^k)_{k=1}^{2^n}$ by

$$v_n(A) = \sum_{I_n^k \subseteq A} \left| T\mathbf{1}_{I_n^k} \right|.$$

Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Observe that for each $A \in \mathcal{F}$ the values $v_n(A)$ are well defined for sufficiently large n. Since, by the Monotone Convergence Theorem, $v_n(A) \le v_{n+1}(A) \le |T|\mathbf{1}_A$ for all $n \in \mathbb{N}$, for every $A \in \mathcal{F}$ there exists a limit in the norm of L_1

$$v(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} v_n(A).$$

Now we show that for almost all $t \in [0,1]$ we have

$$g_{n+1}(t) - v_{n+1}\big([0,1]\big)(t) \le g_n(t) - v_n\big([0,1]\big)(t). \tag{7.66}$$

Indeed, by the definition of g_{n+1}, for almost all t, $g_{n+1}(t) = |(T\mathbf{1}_{I_{n+1}^j})(t)|$ for some $j \in \{1,\dots,2^{n+1}\}$. Let $i \in \{1,\dots,2^n\}$ be such that $I_{n+1}^j \subset I_n^i$. Then

$$\begin{aligned} g_{n+1}(t) - g_n(t) &\le \left|T\mathbf{1}_{I_{n+1}^j}(t)\right| - \left|T\mathbf{1}_{I_n^i}(t)\right| \\ &= v_{n+1}\big(I_{n+1}^j\big)(t) - v_n\big(I_n^i\big)(t) \\ &\le v_{n+1}\big(I_n^i\big)(t) - v_n\big(I_n^i\big)(t) \\ &\le v_{n+1}\big([0,1]\big)(t) - v_n\big([0,1]\big)(t). \end{aligned}$$

Thus, (7.66) is proved. Hence the decreasing sequence $g_n(t) - v_n([0,1])(t)$ has a (finite or $-\infty$) limit a.e. Since the limit $\lim_{n\to\infty} v_n([0,1])(t) = v([0,1])(t)$ is finite and $g_n(t) \ge 0$ a.e., the limit $\lim_{n\to\infty}(g_n(t) - v_n([0,1])(t))$ is finite a.e. Hence, the limit $g(t) = \lim_{n\to\infty} g_n(t)$ exists a.e. on $[0,1]$. Furthermore,

$$0 \le g_n \le v_n\big([0,1]\big) \le v\big([0,1]\big) \in L_1.$$

Therefore, by the Dominated Convergence Theorem, $g = \lim_{n\to\infty} g_n$ in the L_1-norm and $g \in L_1$. $\square$

Definition 7.43. An operator $T \in \mathcal{L}(L_1)$ is called *λ-narrow* if $\lambda_T = 0$.

Pseudonarrow operators and the main result

Recall that an operator $T \in \mathcal{L}(L_1)$ is called pseudonarrow if for every d.p.o. $S \in \mathcal{L}(L_1)$ the inequalities $0 \le S \le |T|$ imply $S = 0$, equivalently, if for every lattice homomorphism $S \in \mathcal{L}(L_1)$ the inequality $S \le |T|$ implies $S = 0$ (cf. Definition 1.32).

Theorem 1.33 applied to the case of $E = F = L_1$ implies the following representation theorem.

Corollary 7.44. *The sets $L_{pe}(L_1)$ of all pseudo-embeddings and $L_{pn}(L_1)$ of all pseudonarrow operators on L_1 are mutually complemented bands. Hence, every operator $T \in \mathcal{L}(L_1)$ has a unique representation in the form $T = T_{pe} + T_{pn}$ where T_{pe} is a pseudo-embedding and T_{pn} is pseudonarrow. Moreover,* $\max\{\|T_{pe}\|, \|T_{pn}\|\} \le \|T\|$.

We will prove that for operators on L_1 the notions of a narrow and a pseudonarrow operators coincide.

Theorem 7.45. *For an operator $T \in \mathcal{L}(L_1)$ the following conditions are equivalent:*

(i) *T is narrow.*

(ii) *T is pseudonarrow.*

(iii) *T is λ-narrow.*

(iv) *For each $A \in \Sigma$ the restriction $T|_{L_1(A)}$ is not an into isomorphism.*

The equivalence (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) can be deduced from Kalton's paper [66]. The implication (iv) $\Rightarrow$ (iii) is older and due to Enflo and Starbird [37]. Finally, the equivalence of (i) with all other conditions was established by Rosenthal in [128].

We will follow the following scheme in our proof:

$$\begin{array}{ccc} & \text{(iv)} & \\ & \updownarrow & \\ & \text{(i)} & \\ \swarrow & & \nwarrow \\ \text{(ii)} & \longrightarrow & \text{(iii)} \end{array}$$

Once we prove the equivalence (i) $\leftrightarrow$ (ii), we obtain the equality $L_{pn}(L_1) = N(L_1)$, and thus Corollary 7.44 implies the following.

Theorem 7.46. *The sets $L_{pe}(L_1)$ of all pseudo-embeddings and $N(L_1)$ of all narrow operators on L_1 are mutually complemented bands (in particular, since $N(L_1)$ is a band in $\mathcal{L}(L_1)$, a sum of two narrow operatores on L_1 is narrow). Hence, every operator $T \in \mathcal{L}(L_1)$ has a unique representation in the form $T = T_{pe} + T_n$ where T_{pe} is a pseudo-embedding and T_n is narrow. Moreover,* $\max\{\|T_{pe}\|, \|T_n\|\} \le \|T\|$.

Note that the last inequality in Theorem 7.46 follows from Proposition 1.30.

Therefore, first we prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and then using Theorem 7.46 we prove the implication (iv) $\Rightarrow$ (i) (the implication (i) $\Rightarrow$ (iv) is obvious).

Proof of the implication (i) $\Rightarrow$ (ii). Let $T \in \mathcal{L}(L_1)$ be narrow. By Corollary 7.44, we may write $T = T_{pe} + T_{pn}$, where T_{pe} is a pseudo-embedding and T_{pn} is pseudonarrow. Suppose on the contrary, that $T_{pe} \neq 0$. Then, by Theorem 7.39, there exists $A \in \Sigma^+$ such that $S_{pe} = T_{pe}|_{L_1(A)}$ is an isomorphic embedding with $d = \|S_{pe}\| \cdot \|S_{pe}^{-1}\| < 2$. Then for $S = T|_{L_1(A)}$ and $S_{pn} = T_{pn}\big|_{L_1(A)}$ we have that $S = S_{pe} + S_{pn}$ with S narrow, S_{pe} pseudo-embedding and S_{pn} pseudonarrow. By Proposition 1.30, $\|S_{pn}\| \le \|S\|$. On the other hand, by Theorem 6.3, $\|S_{pn}\| = \|S_{pe} - S\| \ge \|S\| + \|S_{pe}\|\left(\frac{2}{d} - 1\right) > \|S\|$, a contradiction. □

To prove the implication (ii) $\Rightarrow$ (iii) we need some lemmas (this proof was kindly communicated to us by Mykhaylyuk).

Lemma 7.47. *Let* $0 \le a < b \le 1$, $T \in \mathcal{L}(L_1[a,b], L_1)$ *and* $\varepsilon > 0$. *Then there exists a nonempty dyadic interval* $I \subseteq [0,1]$ *such that for any decomposition* $I = \bigsqcup_{k=1}^n I_k$ *into arbitrary (not necessary dyadic) intervals we have*

$$\Big\| \sup_{1\le k\le n} |T|\,\mathbf{1}_{I_k} - \sup_{1\le k\le n} \big|T\mathbf{1}_{I_k}\big| \Big\| \le \varepsilon \cdot \mu(I)\,. \tag{7.67}$$

Proof. Let $\delta = \varepsilon/(3\|T\|)$ (if $T = 0$ then the assertion of the lemma is trivial). By Theorem 7.31, there exists $A \in \Sigma([a,b])^+$ so that $\|Tx\| \ge \|T\|(1-\delta)\|x\|$ for each $x \in L_1^+(A)$. Then, for each $x \in L_1^+(A)$,

$$\big\|\, |T|\,x - |Tx|\,\big\| = \big\|\,|T|\,x\big\| - \|Tx\| \le \|T\|\|x\| - \|T\|(1-\delta)\|x\| = \delta\|T\|\|x\|\,.$$

Choose a nonempty dyadic interval $I \subseteq [a,b]$ so that $\mu(I \setminus A) \le \delta\mu(I)$. Let $I = \bigsqcup_{k=1}^n I_k$ be a decomposition into arbitrary intervals. Then

$$\begin{aligned}
&\Big\| \sup_{1\le k\le n} |T|\,\mathbf{1}_{I_k} - \sup_{1\le k\le n} \big|T\mathbf{1}_{I_k}\big| \Big\| \\
&\le \sum_{k=1}^n \Big\| |T|\,\mathbf{1}_{I_k} - \big|T\mathbf{1}_{I_k}\big| \Big\| \\
&\le \sum_{k=1}^n \Big\| |T|\,\mathbf{1}_{I_k\cap A} - \big|T\mathbf{1}_{I_k\cap A}\big| \Big\| + \sum_{k=1}^n \Big\| |T|\,\mathbf{1}_{I_k\setminus A} - \big|T\mathbf{1}_{I_k\setminus A}\big| \Big\| \\
&\le \delta\|T\|\mu(I\cap A) + \|T\|\mu(I\setminus A) + \|T\|\mu(I\setminus A) \\
&< 3\delta\|T\|\mu(I) = \varepsilon\mu(I)\,.
\end{aligned}$$

□

Lemma 7.48. *Let $T \in \mathcal{L}(L_1)$. Then $\lambda_T = 0$ if and only if $\lambda_{|T|} = 0$.*

Proof. Suppose that $\lambda_T = 0$ and let $\varepsilon > 0$. By Lemma 7.47, we choose a finite system $(I_j)_1^s$ of disjoint dyadic intervals such that for each j and each decomposition $I_j = \bigsqcup_{k=1}^n I_{j,k}$ into arbitrary intervals the corresponding inequality of the type (7.67) is satisfied and such that $\mu([0,1] \setminus I) < \varepsilon$, where $I = \bigsqcup_{j=1}^s I_j$. Choose m so large that for each $j = 1, \dots, s$, and for the intervals $J_i = [(i-1) \cdot 2^{-m}, i \cdot 2^{-m})$, either $J_i \subseteq I_j$ or $J_i \cap I_j = \emptyset$. Then

$$\begin{aligned}
&\Big\| \sup_{1 \le i \le 2^m} |T| \mathbf{1}_{J_i} - \sup_{1 \le i \le 2^m} |T \mathbf{1}_{J_i}| \Big\| \\
&\le \sum_{j=1}^s \Big\| \sup_{J_i \subseteq I_j} |T| \mathbf{1}_{J_i} - \sup_{J_i \subseteq I_j} |T \mathbf{1}_{J_i}| \Big\| + \sum_{i=1}^{2^m} \big\| |T| \mathbf{1}_{J_i \setminus I} \big\| + \sum_{i=1}^{2^m} \big\| T \mathbf{1}_{J_i \setminus I} \big\| \\
&\le \sum_{j=1}^s \varepsilon \mu(I_j) + 2\|T\| \sum_{i=1}^{2^m} \mu(J_i \setminus I) \\
&\le \varepsilon \mu(I) + 2\|T\| \mu\big([0,1] \setminus I\big) < \varepsilon \left(1 + 2\|T\|\right).
\end{aligned}$$

Thus, $\lambda_{|T|} = 0$. The converse implication is trivial. □

Lemma 7.49. *Let $(A_{k,i})_{k=0,i=1}^{\infty \ \ 2^k}$ be a family of measurable subsets of $[0,1]$ such that*

$$A_{k+1,2i-1} \cup A_{k+1,2i} \subseteq A_{k,i} \tag{7.68}$$

and

$$\bigcup_{i=1}^{2^k} A_{k,i} = A_{0,1} \tag{7.69}$$

for all values of indices. Then there exists a family $(B_{k,i})_{k=0,i=1}^{\infty \ \ 2^k}$ of sets $B_{k,l} \in \Sigma(A_{k,l})$ such that $B_{0,1} = A_{0,1}$ and $B_{k,i} = B_{k+1,2i-1} \sqcup B_{k+1,2i}$ for each k, i.

Proof. For each $k = 0, 1, \dots$ and $i = 1, \dots, 2^k$, let

$$C_{k,i} = \bigcap_{r=0}^{\infty} \Big(\bigcup_{j=1}^{2^r} A_{k+r, 2^r(i-1)+j} \Big).$$

Observe that (7.69) implies $C_{0,1} = A_{0,1}$ and (7.68) implies $C_{k+1,2i-1} \cup C_{k+1,2i} \subseteq C_{k,i}$ for each k, i. Moreover, $C_{k,i} \subseteq A_{k,i}$. Put $B_{0,1} = C_{0,1}$ and inductively on k, $B_{k+1,2i-1} = C_{k+1,2i-1} \cap B_{k,i}$ and $B_{k+1,2i} = B_{k,i} \setminus B_{k+1,2i-1}$. □

Proof of the implication (ii) $\Rightarrow$ (iii). By Lemma 7.48, it is enough to consider only positive operators. Let $T \in \mathcal{L}(L_1)^+$ with $\lambda_T \neq 0$. It is enough to prove that there exists $S \in L_{pe}(L_1)$ such that $0 < S \le T$.

Choose $\varepsilon > 0$ so that $\mu(A_{0,1}) > 0$, where $A_{0,1} = \{t : \lambda_T(t) \geq \varepsilon\}$ and set for $k = 1, 2, \ldots,\ \ i = 1, \ldots, 2^k$,

$$A_{k,i} = \left\{ t \in A_{0,1} : T\mathbf{1}_{\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right)}(t) \geq \varepsilon \right\}.$$

Since $T \geq 0$, the family $(A_{k,i})_{k=0,i=1}^{\infty\ \ 2^k}$ satisfies (7.68) and (7.69). By Lemma 7.49, there exists a family $B_{k,i} = B_{k+1,2i-1} \sqcup B_{k+1,2i}$ satisfying the claims of the lemma. Define an operator $S \in \mathcal{L}(L_1)$ by setting $S\mathbf{1}_{[\frac{i-1}{2^k}, \frac{i}{2^k})} = \varepsilon \mathbf{1}_{B_{k,i}}$ for each $k = 1, 2, \ldots$ and $i = 1, \ldots, 2^k$. Since $B_{k,i} \subseteq A_{k,i}$, we get that $S\mathbf{1}_{[\frac{i-1}{2^k}, \frac{i}{2^k})} \leq T\mathbf{1}_{[\frac{i-1}{2^k}, \frac{i}{2^k})}$. Then S is well defined and $S \leq T$. Since $B_{k+1,2i-1} \cap B_{k+1,2i} = \emptyset$, we obtain that S is a d.p.o. □

For the proof of implication (iii) ⇒ (i) we need the following lemma.

Lemma 7.50. *Let (Ω, Σ, μ) be a finite atomless measure space, $z_1, \ldots, z_{2n} \in L_1(\mu)$, $K = \sum_{i=1}^{2n} \|z_i\|$, $z_0 = \bigvee_{i=1}^{2n} z_i$ and $\alpha = \|z_0\|$. Then there exists a permutation $\tau : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ such that*

$$\Big\| \sum_{i=1}^{2n} (-1)^i z_{\tau(i)} \Big\| \leq \sqrt{2\alpha K}\,.$$

Proof. Let $\delta = \sqrt{2\alpha K}$ and suppose that for each permutation τ we have

$$\Big\| \sum_{i=1}^{2n} (-1)^i z_{\tau(i)} \Big\| > \delta\,.$$

Denote by (r_i) the Rademacher system. Then for each $s \in [0, 1]$, there exists a permutation τ such that

$$\sum_{i=1}^{n} r_i(s)(z_i - z_{n+i}) = \sum_{i=1}^{2n} (-1)^i z_{\tau(i)}\,.$$

Hence for each $s \in [0, 1]$ we have

$$\int_{[0,1]} \Big| \sum_{i=1}^{n} r_i(s)\big(z_i(t) - z_{n+i}(t)\big) \Big|\, \mathrm{d}\mu(t) > \delta\,.$$

Thus, using the Khintchine and the Hölder inequalities, we obtain

$$
\begin{aligned}
\delta &= \int_{[0,1]} \delta\, ds < \int_{[0,1]} \Big(\int_{[0,1]} \Big| \sum_{i=1}^{n} r_i(s)\big(z_i(t) - z_{n+i}(t)\big) \Big| \,\mathrm{d}\mu(t) \Big) \,\mathrm{d}\mu(s) \\
&= \int_{[0,1]} \Big(\int_{[0,1]} \Big| \sum_{i=1}^{n} r_i(s)\big(z_i(t) - z_{n+i}(t)\big) \Big| \,\mathrm{d}\mu(s) \Big) \,\mathrm{d}\mu(t) \\
&\le \int_{[0,1]} \Big(\int_{[0,1]} \Big| \sum_{i=1}^{n} r_i(s) z_i(t) \Big| \,\mathrm{d}\mu(s) + \int_{[0,1]} \Big| \sum_{i=1}^{n} r_i(s) z_{n+i}(t) \Big| \,\mathrm{d}\mu(s) \Big) \,\mathrm{d}\mu(t) \\
&\le \int_{[0,1]} \Big(\sum_{i=1}^{n} z_i^2(t) \Big)^{\frac{1}{2}} \mathrm{d}\mu(t) + \int_{[0,1]} \Big(\sum_{i=1}^{n} z_{n+i}^2(t) \Big)^{\frac{1}{2}} \mathrm{d}\mu(t) \\
&\le \int_{[0,1]} \Big(\bigvee_{i=1}^{n} |z_i(t)| \sum_{i=1}^{n} |z_i(t)| \Big)^{\frac{1}{2}} \mathrm{d}\mu(t) \\
&\quad + \int_{[0,1]} \Big(\bigvee_{i=1}^{n} |z_{n+i}(t)| \sum_{i=1}^{n} |z_{n+i}(t)| \Big)^{\frac{1}{2}} \mathrm{d}\mu(t) \\
&\le \Big(\int_{[0,1]} \bigvee_{i=1}^{n} |z_i(t)| \,\mathrm{d}\mu(t) \Big)^{\frac{1}{2}} \cdot \Big(\int_{[0,1]} \sum_{i=1}^{n} |z_i(t)| \,\mathrm{d}\mu(t) \Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{[0,1]} \bigvee_{i=1}^{n} |z_{n+i}(t)| \,\mathrm{d}\mu(t) \Big)^{\frac{1}{2}} \cdot \Big(\int_{[0,1]} \sum_{i=1}^{n} |z_{n+i}(t)| \,\mathrm{d}\mu(t) \Big)^{\frac{1}{2}} \\
&\le \sqrt{\alpha} \left(\sqrt{\sum_{i=1}^{n} \|z_i\|} + \sqrt{\sum_{i=n+1}^{2n} \|z_i\|} \right) \le \sqrt{\alpha} \sqrt{2 \sum_{i=1}^{2n} \|z_i\|} = \delta \,,
\end{aligned}
$$

a contradiction. □

Proof of the implication (iii) ⇒ (i). Let $T \in \mathcal{L}(L_1)$ be λ-narrow. We assume that $T \neq 0$, otherwise there is nothing to prove. Fix any $\varepsilon > 0$ and any $B \in \Sigma$ of the form $B = \bigcup_{i=1}^{2n} I_m^{k_i}$ where $1 \le k_1 < \ldots < k_{2n} \le 2^m$ and $I_m^i = [2^{-m}(i-1), 2^{-m} i)$. Since $\lambda_T = 0$ and since B can also be represented in the similar form with a larger value of m, we may and do assume that m is so large that

$$
\alpha \stackrel{\text{def}}{=} \Big\| \max_{1 \le i \le 2n} \big| T \mathbf{1}_{I_m^{k_i}} \big| \Big\| \le \Big\| \max_{1 \le k \le 2^m} \big| T \mathbf{1}_{I_m^{k}} \big| \Big\| \le \frac{\varepsilon^2}{2\|T\|\mu(A)} \,.
$$

Let $z_i = T\mathbf{1}_{I_m^{k_i}}$ for $i = 1, \ldots, 2n$. Then $K \stackrel{\text{def}}{=} \sum_{i=1}^{2n} \|z_i\| \le \|T\|\mu(A)$. By Lemma 7.50 there exists a permutation $\tau : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ with

$$
\Big\| \sum_{i=1}^{2n} (-1)^i z_{\tau(i)} \Big\| \le \sqrt{2\alpha K} \le \varepsilon \,.
$$

For $x = \sum_{i=1}^{2n}(-1)^i \mathbf{1}_{I_m^{\tau(k_i)}}$, we obtain that $x^2 = \mathbf{1}_B$, $\int_{[0,1]} x \, \mathrm{d}\mu = 0$ and $\|Tx\| < \varepsilon$. By Lemma 1.12, T is narrow. □

Upon completion of proving the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (i), we have proved Theorem 7.46.

Proof of the implication (iv) ⇒ (i). Let $T \in \mathcal{L}(L_1)$, and assume that for every $A \in \Sigma$, $T|_{L_1(A)}$ is not an into isomorphism. By Theorem 7.46, we represent $T = T_{pe} + T_n$, where T_{pe} is a pseudo-embedding and T_n is narrow. Our goal is to prove that $T_{pe} = 0$.

Suppose, on the contrary, that $T_{pe} \neq 0$. Let $\varepsilon = \|T\|/2$. By uniqueness of the representation $T = T_{pe} + T_n$, $T \neq 0$, and hence, $\varepsilon > 0$. Choose $\delta > 0$ so that

$$\big(\|T\| - \delta\big)\Big(\frac{2}{1+\delta} - 1\Big) \geq \|T\| - \varepsilon$$

and

$$\eta \overset{\text{def}}{=} \frac{\|T\| - \delta}{1+\delta} - \frac{\|T\|}{2} > 0\,.$$

By Theorem 7.39, there exists $A \in \Sigma$ so that $S_{pe} \overset{\text{def}}{=} T_{pe}|_{L_1(A)}$ is an into isomorphism with $\|S_{pe}\| \geq \|T\| - \delta$ and $\|S_{pe}\|\|S_{pe}^{-1}\| < 1 + \delta$. Then setting $S = T|_{L_1(A)}$ and $S_n = T|_{L_1(A)}$, we obtain the decomposition $S = S_{pe} + S_n$, where S_{pe} is an into isomorphism, S_n is narrow and S is not an into isomorphism. By Theorem 6.3,

$$\begin{aligned}\|T\| \geq \|S\| \geq \|S_n\| + \|S_{pe}\|\Big(\frac{2}{1+\delta} - 1\Big)\\ \geq \|S_n\| + \big(\|T\| - \delta\big)\Big(\frac{2}{1+\delta} - 1\Big)\\ \geq \|S_n\| + \|T\| - \varepsilon\,.\end{aligned}$$

Thus, $\|S_n\| \leq \varepsilon$. Hence, for each $x \in L_1(A)$, we have

$$\begin{aligned}\|Tx\| = \|Sx\| \geq \|S_{pe}x\| - \|S_n x\| \geq \big\|S_{pe}^{-1}\big\|^{-1}\|x\| - \varepsilon\|x\|\\ \geq \left(\frac{\|S_{pe}\|}{1+\delta} - \varepsilon\right)\|x\|\\ \geq \left(\frac{\|T\| - \delta}{1+\delta} - \frac{\|T\|}{2}\right)\|x\| = \eta\|x\|\,.\end{aligned}$$

This is impossible since S is not an into isomorphism. □

Open problems

By Theorem 7.46, a sum of two narrow operators from L_1 to L_1 is narrow. Very recently[1] appeared the first example of a Banach space X and two narrow operators from L_1 to X with a non-narrow sum.

[1] V.Mykhaylyuk, M.Popov. On sums of narrow operators on Köthe function spaces. Preprint.

Open problem 7.51. Characterize Banach spaces X for which a sum of two narrow operators in $\mathcal{L}(L_1, X)$ is narrow.

Theorem 7.30 characterizes when an operator $T \in \mathcal{L}(L_1)$ is narrow. An analogous result does not hold for $p > 2$. To see this, consider the composition $SJ \in \mathcal{L}(L_p)$, where $J : L_p \to L_2$ is the identity embedding and $S : L_2 \to L_p$ is any isomorphic embedding. However we do not know whether a similar characterization is true for $1 < p \leq 2$.

Open problem 7.52. Suppose an operator $T \in \mathcal{L}(L_p)$, $1 < p \leq 2$, is such that for every $A \in \Sigma^+$ the restriction $T|_{L_p(A)}$ is not an isomorphic embedding. Does it follow that T is narrow?

In Section 11.2 we present a partial answer to this problem.

In Chapter 10 we state a general lattice version of Open problem 7.52, see Open problem 10.46.

As a corollary of Theorem 7.30 we obtain the following sufficient condition for subspaces of L_1 to be rich.

Corollary 7.53. *Let X be a subspace of L_1 such that the quotient space L_1/X isomorphically embeds in L_1. If $\rho(L_1(A), X) = 0$ for every $A \in \Sigma^+$, then X is rich (here ρ is a function defined by* (6.60)*).*

Talagrand's Theorem (see Theorem 8.24 below) shows that the isomorphic embeddability condition of L_1/X in L_1 is essential in Corollary 7.53. Indeed, let X be a subspace of L_1 such that neither X nor L_1/X contains a copy of L_1. Since L_1 does not embed in X, $\rho(L_1(A), X) = 0$ for every $A \in \Sigma^+$ (indeed, if $\rho(L_1(A), X) \neq 0$ then the restriction $\tau_{L_1(A)}$ of the quotient map $\tau : L_1 \to L_1/X$ is an isomorphic embedding). By Corollary 2.23, since L_1 does not embed in L_1/X, X is not rich.

Open problem 7.54. Let E be an r.i. Banach space on $[0, 1]$, $E \neq L_1$. Let X be a subspace of E such that $\rho(L_1(A), X) = 0$ for every $A \in \Sigma^+$. Must X be rich?

7.3 Johnson–Maurey–Schechtman–Tzafriri's theorem on narrowness of non-Enflo operators on L_p for $1 < p < 2$

This section is devoted to a proof that Problem 7.1(c) has an affirmative answer for L_p when $1 < p < 2$. This result was first explicitly stated by Bourgain [19, Theorem 4.12, item 2] as a result that can be deduced from the proof of a related result in Johnson, Maurey, Schechtman and Tzafriri's book [49].

Theorem 7.55. *Let $1 < p < 2$. Then every non-Enflo operator $T \in \mathcal{L}(L_p)$ is narrow.*

The assertion of Theorem 7.55 is evidently true for $p = 2$, but false for $p > 2$ due to the following example.

Example 7.56. Let $p > 2$ and $T = S \circ J$ where $J : L_p \to L_2$ is the inclusion embedding and $S : L_2 \to L_p$ is an isomorphic embedding. Then T is not Enflo and not narrow.

Recently Dosev, Johnson and Schechtman [32] proved the following result about the detailed structure of nonnarrow operators on L_p, $1 < p < 2$.

Theorem 7.57. *For each* $1 < p < 2$ *there is a constant* K_p *such that if* T *from* $L_p[0,1]$ *to* L_p *is a nonnarrow operator (and in particular if it is an isomorphism), then there is a* K_p*-complemented subspace* X *of* L_p *which is* K_p*-isomorphic to* L_p *and such that* $T\big|_X$ *is a* K_p*–isomorphism and* $T(X)$ *is* K_p *complemented in* L_p*.*

Moreover, if we consider L_p *with the norm* $|||x|||_p = \|S(x)\|_p$ *(with* S *being the square function with respect to the Haar system, see Definition* 7.72 *below) then, for each* $\varepsilon > 0$*, there exists a subspace* X *of* L_p *which is* $(1+\varepsilon)$*-isomorphic to* L_p *and such that some multiple of* $T\big|_X$ *is a* $(1+\varepsilon)$*-isomorphism (and* X *and* $T(X)$ *are* K_p *complemented in* L_p*).*

It is not known whether an analog of Theorem 7.30 is true for $1 < p \le 2$.

In this section we present a proof of Theorem 7.55 which closely follows the proof of Theorem 9.1 in [49] with all necessary adjustments due to the fact that we work with a weaker hypothesis.

Somewhat narrow operators and generalized sign-embeddings

Definition 7.58. Let E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ), and let X be a Banach space. An operator $T \in \mathcal{L}(E, X)$ is called *somewhat narrow* if for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a set $B \in \Sigma^+(A)$ and a sign x on B such that $\|Tx\| < \varepsilon\|x\|$.

Obviously, each narrow operator is somewhat narrow. The inclusion embedding $J : L_p \to L_r$ with $1 \le r < p < \infty$ is an example of a somewhat narrow operator which is not narrow.

We split the proof of Theorem 7.55 into two parts: Theorems 7.59 and 7.60.

Theorem 7.59. *Let* $1 \le p \le 2$*. Then every somewhat narrow operator* $T \in \mathcal{L}(L_p)$ *is narrow.*

Theorem 7.60. *Let* $1 < p \le 2$*. Then every non-Enflo operator on* L_p *is somewhat narrow.*

Theorem 7.59 is not true for $p > 2$ as Example 7.56 shows. We do not know whether Theorem 7.60 is true when $p > 2$.

Observe that an operator $T \in \mathcal{L}(E, X)$ is not somewhat narrow if and only if there exist $A \in \Sigma^+$ and $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for every sign $x \in E$ with $\operatorname{supp} x \subseteq A$.

Definition 7.61. We say that an operator $T \in \mathcal{L}(L_p, X)$ is a *generalized sign-embedding* if $\|Tx\| \geq \delta\|x\|$ for some $\delta > 0$ and every sign $x \in E$.

We remark that in some papers (see [126, 127]) Rosenthal studied a closely related notion of a sign-embedding defined on L_1, but in his definition an additional assumption of injectivity of T was required. Formally, this is not the same, and there is an operator on L_1 that is bounded from below at signs and is not injective (and even has a kernel isomorphic to L_1), see Section 8.1. However, if an operator on L_1 is bounded from below at signs then there exists $A \in \Sigma^+$ such that the restriction $T|_{L_1(A)}$ is injective, and hence is a sign-embedding in the sense of Rosenthal. Using this notion, Theorem 7.60 can be equivalently reformulated as follows.

Theorem 7.62. *Let $1 < p \leq 2$. Then every generalized sign-embedding on L_p is an Enflo operator.*

Thus, to prove Theorem 7.55, it is enough to prove Theorems 7.59 and 7.62.

A proof of Theorem 7.59 and its generalization for operators from L_1 to any Banach space

The following lemma will be used below in different contexts.

Lemma 7.63. *Let (Ω, Σ, μ) be a finite atomless measure space and $1 \leq p < \infty$. Then we have the following:*

(a) *For each $x, y \in L_p(\mu)$ we have*

$$\min\{\|x+y\|, \|x-y\|\} \leq (\|x\|^p + \|y\|^p)^{1/p} \quad \text{if } 1 \leq p \leq 2, \quad \text{and}$$

$$(\|x\|^p + \|y\|^p)^{1/p} \leq \max\{\|x+y\|, \|x-y\|\} \quad \text{if } 2 \leq p < \infty.$$

(b) *For each $n \in \mathbb{N}$ and each vectors $(z_k)_{k=1}^n$ in $L_p(\mu)$ there is a collection of sign numbers $(\theta_k)_{k=1}^n$ such that*

$$\Big\|\sum_{k=1}^n \theta_k z_k\Big\| \leq \Big(\sum_{k=1}^n \|z_k\|^p\Big)^{1/p} \quad \text{if } 1 \leq p \leq 2, \quad \text{and}$$

$$\Big(\sum_{k=1}^n \|z_k\|^p\Big)^{1/p} \leq \Big\|\sum_{k=1}^n \theta_k z_k\Big\| \quad \text{if } 2 \leq p < \infty.$$

(c) *For any unconditionally convergent series* $\sum_{n=1}^{\infty} x_n$ *in* $L_p(\mu)$ *there is a sequence* $(\theta_n)_{n=1}^{\infty}$ *of sign numbers such that*

$$\Big\|\sum_{n=1}^{\infty} \theta_n x_n\Big\| \le \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p} \quad \textit{if } 1 \le p \le 2,$$

$$\Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p} \le \Big\|\sum_{n=1}^{\infty} \theta_n x_n\Big\| \quad \textit{if } 2 \le p < \infty.$$

Note that Lemma 7.63(b) for the case $1 \le p \le 2$ exactly means that the space $L_p(\mu)$ has infratype p with constant one (see [95]). Lemma 7.63 is a consequence of the Orlicz theorem (see [25, p. 101]). However, one can prove the lemma in an easy way. Indeed, for $p = 1$ inequality (a) follows from the triangle inequality, and for $1 < p < \infty$ it is a consequence of Clarkson's inequality (see, e.g. [25, p. 117–118]) (b) and (c) are consequences of (a).

Proof of Theorem 7.59. Fix any $A \in \Sigma^+$ and $\varepsilon > 0$. To prove that T is narrow it is enough to prove that $\|Tx\| \le \varepsilon\mu(A)^{1/p}$ for some sign x on A.

Assume, for contradiction, that for each sign x on A we have

$$\|Tx\| > \varepsilon\mu(A)^{1/p}.$$

We will construct a transfinite sequence $(A_\alpha)_{\alpha<\omega_1}$ of uncountable length ω_1 of disjoint sets $A_\alpha \in \Sigma^+$, $A_\alpha \subset A$, which will give us the desired contradiction.

By the definition of a somewhat narrow operator, there exist a set $A_0 \in \Sigma^+$, $A_0 \subseteq A$ and a sign x_0 on A_0 such that

$$\|Tx_0\| \le \varepsilon\mu(A_0)^{1/p} .$$

Observe that by our assumption, x_0 cannot be a sign on A, therefore, $\mu(A\setminus A_0) > 0$.

Suppose that for a given ordinal $0 < \beta < \omega_1$ we have constructed a transfinite sequence of disjoint sets $(A_\alpha)_{\alpha<\beta} \subseteq \Sigma^+$, $A_\alpha \subset A$ and a transfinite sequence $(x_\alpha)_{\alpha<\beta}$ of signs x_α on A_α such that

$$\|Tx_\alpha\| \le \varepsilon\mu(A_\alpha)^{1/p} .$$

Let $B = \bigcup_{\alpha<\beta} A_\alpha$. Our goal is to prove that $\mu(A \setminus B) > 0$. Since $(x_\alpha)_{\alpha<\beta}$ is a disjoint sequence in $L_p(\mu)$ with $|x_\alpha| \le 1$ a.e., we have that the series $\sum_{\alpha<\beta} x_\alpha$ is unconditionally convergent, and so is the series $\sum_{\alpha<\beta} Tx_\alpha$. By Lemma 7.63(b), there exist sign numbers $\theta_\alpha = \pm 1$, $\alpha < \beta$ so that

$$\begin{aligned}\Big\|\sum_{\alpha<\beta} \theta_\alpha T x_\alpha\Big\| &\le \Big(\sum_{\alpha<\beta} \|Tx_\alpha\|^p\Big)^{1/p} \\ &\le \Big(\sum_{\alpha<\beta} \varepsilon^p \mu(A_\alpha)\Big)^{1/p} = \varepsilon\mu(B)^{1/p}.\end{aligned} \tag{7.70}$$

Observe that $x = \sum_{\alpha<\beta} \theta_\alpha x_\alpha$ is a sign on B and, by (7.70), $\|Tx\| \le \varepsilon\mu(B)^{1/p}$. By our assumption, x cannot be a sign on A and hence $\mu(A \setminus B) > 0$. Using the definition of a somewhat narrow operator, there exists $A_\beta \in \Sigma^+$, $A_\beta \subseteq A$ and a sign x_β on A_β such that $\|Tx_\beta\| \le \varepsilon\mu(A_\beta)^{1/p}$. Thus, the recursive construction is done. □

When $p = 1$, the assertion of Theorem 7.59 holds for operators valued in any Banach space (this statement was mentioned by Ghoussoub and Rosenthal in [44]).

Theorem 7.64. *Let X be a Banach space. Then every somewhat narrow operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Proof. Fix any $A \in \Sigma^+$ and $\varepsilon > 0$. Denote by $\mathcal{A}$ the set of all $B \in \Sigma^+(A)$ for which there exists a sign x on B with $\|Tx\| \le \varepsilon\|x\|$. By Zorn's Lemma we deduce that there exists a maximal collection $\mathcal{M}$ of disjoint elements of $\mathcal{A}$. Since each element of $\mathcal{M}$ has support of positive measure, $\mathcal{M}$ is at most countable: $\mathcal{M} = \{B_i\}_{i\in I}$ with I finite or countable. Let

$$A_0 = \bigcup \mathcal{M} = \bigcup_{i\in I} B_i \,.$$

We claim that $A_0 \in \mathcal{A}$. Indeed, for each $i \in I$ choose a sign x_i on B_i with $\|Tx_i\| \le \varepsilon\|x_i\|$. Then $x = \sum_{i\in I} x_i$ is a sign on A_0 and

$$\|Tx\| \le \sum_{i\in I} \|Tx_i\| \le \varepsilon \sum_{i\in I} \|x_i\| = \varepsilon\|x\| \,.$$

Thus, $A_0 \in \mathcal{A}$. Now we prove that $\mu(A \setminus A_0) = 0$. Supposing the contrary, we would obtain by the theorem assumptions that there exist $B \in \Sigma^+(A \setminus A_0)$ and a sign x at B with $\|Tx\| \le \varepsilon\|x\|$. This contradicts the maximality of $\mathcal{M}$. Thus, the sets A and A_0 coincide a.e.

We have proved the following statement: for each $A \in \Sigma^+$ and $\varepsilon > 0$ there exists a sign x supported on A with $\|Tx\| \le \varepsilon$. By Proposition 1.9, T is narrow. □

A proof of Theorem 7.62. Part 1. Preliminary arguments and an outline

First we recall the Khintchine inequality which will be the starting point of the proof. Let (r_n) be the Rademacher system on [0, 1]. Then for every $p \in [1, +\infty)$ there are constants $0 < A_p \le B_p < \infty$ such that

$$A_p\Big(\sum_{i=1}^n |a_k|^2\Big)^{1/2} \le \left(\int_{[0,1]}\Big|\sum_{i=1}^n a_i r_i(t)\Big|^p dt\right)^{1/p} \le B_p\Big(\sum_{i=1}^n |a_k|^2\Big)^{1/2}$$

for every $n \in \mathbb{N}$ and every choice of scalars $(a_i)_{i=1}^n$, see [79, p. 66].

The following statement, which is due to Maurey (see [80, p. 50] for a general setting of q-concave Banach lattices) is the main tool of the proof.

Lemma 7.65. *Let* $(x_i)_{i=1}^n$ *be a* K*-unconditional basic sequence in* L_p *with* $1 \le p < \infty$. *Then*

$$A_1 K^{-1} \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2} \Big\| \le \Big\| \sum_{i=1}^n x_i \Big\| \le B_p K \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2} \Big\| , \tag{7.71}$$

where A_1 *and* B_p *are constants from Khintchine's inequality.*

Proof. By unconditionality, the triangle inequality and the left-hand side Khintchine's inequality for $p = 1$, respectively, we have

$$\Big\| \sum_{i=1}^n x_i \Big\| \ge K^{-1} \int_{[0,1]} \Big\| \sum_{i=1}^n r_i(s) x_i \Big\| ds \ge K^{-1} \Big\| \int_{[0,1]} \Big| \sum_{i=1}^n r_i(s) x_i \Big| ds \Big\|$$
$$\ge A_1 K^{-1} \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2} \Big\| .$$

On the other hand, by unconditionality, Hölder's inequality, Fubini's theorem and the right-hand side of Khintchine's inequality, respectively, we obtain

$$\Big\| \sum_{i=1}^n x_i \Big\| \le K \int_{[0,1]} \Big\| \sum_{i=1}^n r_i(s) x_i \Big\| ds \le K \Big(\int_{[0,1]} \Big\| \sum_{i=1}^n r_i(s) x_i \Big\|^p ds \Big)^{1/p}$$
$$= K \Big\| \Big(\int_{[0,1]} \Big| \sum_{i=1}^n r_i(s) x_i \Big|^p ds \Big) \Big\| \le B_p K \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2} \Big\| . \qquad \square$$

For $n = 0, 1, \dots$ and $i = 1, \dots, 2^n$, let $I_{n,i}$ be the dyadic interval $[2^{-n}(i-1), 2^{-n} i)$. Then we use the following notation for the L_∞-normalized Haar system

$$h_{0,0} = \mathbf{1}_{[0,1]}, \quad h_{n,i} = \mathbf{1}_{I_{n+1,2i-1}} - \mathbf{1}_{I_{n+1,2i}}, \quad n = 0, 1, \dots, \quad i = 1, \dots, 2^n .$$

Observe that $\operatorname{supp} h_{n,i} = I_{n,i}$. The L_p-normalized Haar system in L_p will be denoted by $\{h_{0,0}\} \cup (h_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$, or by $(h_{n,i})$, in short.

The *square function* $S : L_p \to L_p^+$ with respect to the Haar system $(h_{n,i})$ is defined by

$$S\Big(\sum_{(n,i)} a_{n,i} h_{n,i} \Big) = \Big(\sum_{(n,i)} |a_{n,i} h_{n,i}|^2 \Big)^{1/2} . \tag{7.72}$$

Using this notation, we obtain the following consequence of Lemma 7.65.

Corollary 7.66. *For any* $x \in L_p$, $1 < p < \infty$ *we have*

$$A_1 K_p^{-1} \big\| S(x) \big\| \le \|x\| \le B_p K_p \big\| S(x) \big\| , \tag{7.73}$$

where A_1 *and* B_p *are constants from the Khintchine inequality and* K_p *is the unconditional constant of the Haar system in* L_p.

Following [49], a sequence (x_n) in X is said to be *disjointly supported with respect to a basis* (e_i) of X provided that $e_i^*(x_n)e_i^*(x_m) = 0$ for all integers i and $n \neq m$, where (e_i^*) are the biorthogonal functionals to (e_i). We will use the following two simple observations.

(a) Every block basis of $(h_{n,i})$ is disjointly supported with respect to $(h_{n,i})$.

(b) If a sequence (x_n) is disjointly supported with respect to $(h_{n,i})$ then the square function S has the following property:

$$S^2\Big(\sum_n x_n\Big) = \sum_n S^2(x_n). \tag{7.74}$$

The proof of Theorem 7.62 is very long and will be split into several propositions and lemmas. We start from statements of these intermediate results postponing their proofs to the end of this section. This will provide the outline of the proof of Theorem 7.62.

Lemma 7.67. *It is enough to prove Theorem* 7.62 *for an operator T for which the sequence $(Th_{n,i})$ of images of the Haar system is disjointly supported with respect to the Haar system $(h_{n,i})$.*

Lemma 7.68. *Let $T \in \mathcal{L}(L_p)$, $1 < p < 2$ be so that $(Th_{n,i})$ is disjointly supported with respect to the Haar system $(h_{n,i})$ and there exists $\delta > 0$ so that*

$$\|Th\| \geq \delta\|h\|$$

for every sign h. Define

$$v_n = S\Big(\sum_{i=1}^{2^n} Th_{n,i}\Big), \quad n = 0, 1, \ldots . \tag{7.75}$$

Then the following properties are satisfied:

(P1) *There exists $\gamma > 0$ such that $\|v_n\| \geq \gamma$ for each $n = 0, 1, \ldots$.*

(P2) *The sequence (v_n^p) is equi-integrable, i.e. for each $\varepsilon > 0$ there exists $R < \infty$ such that*

$$\int_{\{v_n \geq R\}} v_n^p \, d\mu < \varepsilon$$

for each $n = 0, 1, \ldots$.

(P3) *There exist numbers $R, \eta > 0$ such that*

$$\int_{\{v_n < R\}} v_n^p \, d\mu \geq \eta, \quad n = 0, 1, \ldots . \tag{7.76}$$

For each $n = 0, 1, \ldots$ we define an $L_{p/2}$-valued measure on the algebra $\mathcal{I}_n$ generated by the dyadic intervals $I_{n,i}$ of length 2^{-n} by setting

$$\nu_n(A) = S^2\Big(\sum_{\operatorname{supp} h_{n,i} \subseteq A} Th_{n,i}\Big) \cdot \mathbf{1}_{\{v_n < R\}}, \quad A \in \mathcal{I}_n \,. \tag{7.77}$$

The finite additivity of ν_n on $\mathcal{I}_n$ follows from the fact that the sequence $(Th_{n,i})$ is disjointly supported with respect to $(h_{n,i})$ and (7.74). We denote $\mathcal{E} = \bigcup_{n=0}^{\infty} \mathcal{I}_n$. Observe that for each $A \in \mathcal{E}$ the sequence $(\nu_n(A))_{n=0}^{\infty}$ is uniformly bounded. Indeed, by (7.74)

$$v_n^2 = S^2\Big(\sum_{i=1}^{2^n} Th_{n,i}\Big) = S^2\Big(\sum_{\operatorname{supp} h_{n,i} \subseteq A} Th_{n,i}\Big) + S^2\Big(\sum_{\operatorname{supp} h_{n,i} \not\subseteq A} Th_{n,i}\Big)$$

and hence,

$$S^2\Big(\sum_{\operatorname{supp} h_{n,i} \subseteq A} Th_{n,i}\Big) \leq v_n^2 \,.$$

Multiplying the last inequality by $\mathbf{1}_{\{v_n < R\}}$, we obtain

$$\nu_n(A) \leq v_n^2 \cdot \mathbf{1}_{\{v_n < R\}} \leq R^2 \,. \tag{7.78}$$

Since a bounded set in L_2 is relatively weakly compact, there exists a subsequence $(\nu_{n_k}(A))_{k=1}^{\infty}$ which converges weakly in L_2 to a limit which we denote by $\nu(A)$. By Mazur's theorem, there exist disjoint sets $N_\ell = N_\ell(A)$, $\ell = 1, 2, \ldots$ of integers, and numbers $\alpha_n = \alpha_n(A) \geq 0$, $n = 1, 2, \ldots$ with $\sum_{n \in N_\ell} \alpha_n = 1$ for each $\ell = 1, 2, \ldots$ such that

$$\lim_{\ell \to \infty} \sum_{n \in N_\ell} \alpha_n \nu_n(A) = \nu(A) \,, \tag{7.79}$$

where the convergence is in L_2, and thus, also in L_1, in $L_{p/2}$, and almost everywhere. Using (7.79), one can easily show that ν is a finitely additive measure on $\mathcal{E}$. Observe that (7.78) and (7.79) imply

$$\nu(A) \leq R^2 \quad \text{a.e. on } [0,1] \,. \tag{7.80}$$

Lemma 7.69. *The above defined finitely additive measure ν has the following properties:*

(P4) $\nu([0,1]) \neq 0$.

(P5) *There exists a measurable set $\Omega' \subseteq [0,1]$ and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$*

$$\int_{\Omega'} \max_{1 \leq i \leq 2^n} \nu(I_{n,i}) \,\mathrm{d}\mu \geq \varepsilon,$$

and for every $F \in \mathcal{E}$ *the pointwise convergence in* (7.79)

$$\lim_{\ell\to\infty}\sum_{n\in N_\ell}\alpha_n\nu_n(F)=\nu(F)$$

is uniform on Ω'.

(P6) *There exists a constant* $C > 0$ *so that*

$$\int_{[0,1]}\nu(A)\,\mathrm{d}\mu\le C\mu(A)\,, \tag{7.81}$$

and thus ν *can be extended to an* L_1^+*-valued countably additive measure on* Σ.

The next lemma is a slight modification of [49, Lemma 9.8]. We start with definitions. A *tree* $(F_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ *of sets* is a family of measurable subsets of [0, 1] such that $F_{n,i} = F_{n+1,2i-1}\sqcup F_{n+1,2i}$ and $\mu(F_{n,i}) = 2^{-n}$ for every $n = 0,1,\dots$ and $i = 1,\dots,2^n$. For a constant $C > 0$, a C*-tree over a measure space* $(\Omega,\mathcal{F},\lambda)$ is a collection of sets $(G_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ in $\mathcal{F}$ such that $G_{n,i} = G_{n+1,2i-1}\sqcup G_{n+1,2i}$ and

$$\frac{1}{C2^n}\le\lambda(G_{n,i})\le\frac{C}{2^n}$$

for all $n = 0,1,\dots$ and $i = 1,\dots,2^n$.

Lemma 7.70. *Let* ν *be a measure on* $([0,1],\Sigma)$ *taking values in* $L_1^+(\Omega,\mathcal{F},\lambda)$ *where* λ *is a finite measure. Assume the following:*

(i) *The semivariation of* ν *is absolutely continuous with respect to the Lebesgue measure* μ*, i.e.*

$$\lim_{\mu(A)\to 0}\int_\Omega\nu(A)\,\mathrm{d}\lambda=0.$$

(ii) *There are* $\varepsilon > 0$ *and* $\Omega'\in\mathcal{F}$ *such that for each* $n = 0,1,\dots$ *we have*

$$\int_{\Omega'}\max_{1\le i\le 2^n}\nu(I_{n,i})\,\mathrm{d}\lambda\ge\varepsilon.$$

Then there exist constants $C,\xi > 0$*, a tree* $(F_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ *with* $F_{n,i}\in\mathcal{E}$*, and a* C*-tree* $(G_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ *in* $\mathcal{F}(\Omega')$ *such that for each* $n = 0,1,\dots$ *and* $i = 1,\dots,2^n$

$$\nu\big(F_{n,i}\big)(t)\ge\xi\quad\textit{for all }\ t\in G_{n,i}\,. \tag{7.82}$$

Lemma 7.71. *Using the same notation as in Lemma* 7.70*, there exists a sequence* $(N_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ *of disjoint finite sets of integers such that* $\min N_{m,j} > \min\{\ell : F_{m,j}\in\mathcal{E}_\ell\}$ *and a collection of nonnegative numbers* $\{\beta_n : n\in\bigcup_{m,j}N_{m,j}\}$*, such* *that* $\sum_{n\in N_{m,j}} \beta_n = 1$, *for* $m = 0, 1, \ldots$, $j = 1, \ldots, 2^m$ *and*

$$\sum_{n\in N_{m,j}} \beta_n \nu_n\big(F_{m,j}\big)(t) \ge \frac{\xi}{2} \quad \textit{for all} \ \ t \in G_{m,j}\ . \tag{7.83}$$

We are now ready for the final step of the proof of Theorem 7.62. We define for $m = 0, 1, \ldots,\ j = 1, \ldots, 2^m$,

$$\begin{aligned} \widetilde{h}_{0,0} &= \mathbf{1}, \\ \widetilde{h}_{m,j} &= \sum_{n\in N_{m,j}} \beta_n^{1/2} \sum_{\operatorname{supp} h_{n,i}\subseteq F_{m,j}} h_{n,i}\,, \qquad (7.84)\\ k_{0,0} &= T\widetilde{h}_{0,0} = T\mathbf{1}, \\ k_{m,j} &= T\widetilde{h}_{m,j} = \sum_{n\in N_{m,j}} \beta_n^{1/2} \sum_{\operatorname{supp} h_{n,i}\subseteq F_{m,j}} T h_{n,i}\,. \qquad (7.85) \end{aligned}$$

It suffices to prove that both $(\widetilde{h}_{m,j})_{m,j}$ and $(k_{m,j})_{m,j}$ are equivalent to the Haar system in L_p. When this is established, we see that T acts as an isomorphism on $H = [\widetilde{h}_{m,j}] \subset L_p$, which will end the proof of Theorem 7.62.

The proof that $(k_{m,j})_{m,j}$ is equivalent to the Haar system in L_p follows from the following modification of [49, Proposition 9.6]:

Proposition 7.72. *Let X be an r.i. function space on* $[0,1]$ *whose Boyd indices satisfy* $0 < \beta_X \le \alpha_X < 1$. *Let* $\{k_{0,0}\} \cup (k_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ *be a block basis of some enumeration of the Haar system, and for some* $C > 0$, *let* $(G_{m,j})_{n=0}^{\infty}{}_{j=1}^{2^m}$ *be a C-tree on* $[0,1]$ *such that*

(i) $\{k_{0,0}\} \cup (k_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ *is C-dominated by the Haar system, i.e.*

$$\Big\|\sum_{m,j} a_{m,j} k_{m,j}\Big\| \le C \Big\|\sum_{m,j} a_{m,j} h_{m,j}\Big\|$$

for every sequence $\{a_{0,0}\} \cup (a_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ *of scalars;*

(ii) $\displaystyle\int_{G_{m,j}} S(k_{m,j})\,\mathrm{d}\mu \ge C^{-1} 2^{-m}$, $m = 0, 1, \ldots$, $j = 1, \ldots, 2^m$.

Then the system $\{k_{0,0}\} \cup (k_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ *is equivalent to the Haar system in X.*

We check that the system $\{k_{0,0}\} \cup (k_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ satisfies assumptions (i) and (ii) of Proposition 7.72.

To see (i), let $\{a_{0,0}\} \cup (a_{m,j})_{m=0}^{\infty}{}_{j=1}^{2^m}$ be any scalars. Then

$$\begin{aligned}
\Big\|\sum_{m,j} a_{m,j} k_{m,j}\Big\| &= \Big\|T\Big(\sum_{m,j} a_{m,j} \sum_{n\in N_{m,j}} \beta_n^{1/2} \sum_{\operatorname{supp} h_{n,i}\subseteq F_{m,j}} h_{n,i}\Big)\Big\| \\
&\overset{\text{by (7.65)}}{\le} \|T\| K_p B_p \Big\|\Big(\sum_{m,j} a_{m,j}^2 \sum_{n\in N_{m,j}} \beta_n \sum_{\operatorname{supp} h_{n,i}\subseteq F_{m,j}} h_{n,i}\Big)^{1/2}\Big\| \qquad (7.86)\\
&= \|T\| K_p B_p \Big\|\Big(\sum_{m,j} a_{m,j}^2 \sum_{n\in N_{m,j}} \beta_n \mathbf{1}_{F_{m,j}}\Big)^{1/2}\Big\| \\
&= \|T\| K_p B_p \Big\|\Big(\sum_{m,j} a_{m,j}^2 \mathbf{1}_{F_{m,j}}\Big)^{1/2}\Big\|.
\end{aligned}$$

On the other hand, by (7.65),

$$\begin{aligned}
\Big\|\sum_{m,j} a_{m,j} h_{m,j}\Big\| &\ge A_1 K_p^{-1} \Big\|\Big(\sum_{m,j} a_{m,j}^2 h_{m,j}\Big)^{1/2}\Big\| \\
&= A_1 K_p^{-1} \Big\|\Big(\sum_{m,j} a_{m,j}^2 \mathbf{1}_{I_{m,j}}\Big)^{1/2}\Big\| \qquad (7.87)\\
&= A_1 K_p^{-1} \Big\|\Big(\sum_{m,j} a_{m,j}^2 \mathbf{1}_{F_{m,j}}\Big)^{1/2}\Big\|
\end{aligned}$$

since $(F_{m,j})$ is a tree of disjoint sets with $\mu(F_{m,j}) = \mu(I_{m,j})$, and L_p is an r.i. space. Then (7.86) and (7.87) together give (i).

(ii) By the definitions of $(k_{m,j})$, $(N_{m,j})$ and (β_n) from Lemma 7.71, for all $t \in G_{m,j}$ we have

$$S\big(k_{m,j}\big)(t) \ge \Big(\sum_{n\in N_{m,j}} \beta_n \nu_n\big(F_{m,j}\big)(t)\Big)^{1/2} \ge \sqrt{\frac{\xi}{2}}\,.$$

Since $\mu(G_{m,j}) \ge C_0^{-1} 2^{-m}$, we get (ii).

This completes the proof that $(k_{m,j})_{m,j}$ is equivalent to the Haar system in L_p.

To prove that $(\widetilde{h}_{m,j})_{m,j}$ is equivalent to the Haar system in L_p, we first recall a notion which was introduced in [49].

Definition 7.73. Let $(\widetilde{F}_{m,j})_{n=0}^{\infty}{}_{j=1}^{2^m}$ be a tree of elements of $\mathcal{E}$, and $(\widetilde{N}_{m,j})_{n=0}^{\infty}{}_{j=1}^{2^m}$ be a family of subsets of the integers such that

(i) $\min \widetilde{N}_{m,j} > \min\{\ell : \widetilde{F}_{m,j} \in \mathcal{E}_\ell\}$;

(ii) $\widetilde{N}_{k,j} \cap \widetilde{N}_{m,i} = \emptyset$ whenever $\widetilde{F}_{k,j} \subsetneq \widetilde{F}_{m,i}$.

Let $\widetilde{\beta}_n$, $n \in \bigcup_{m,j} \widetilde{N}_{m,j}$ be reals such that $\sum_{n\in\widetilde{N}_{m,j}} \widetilde{\beta}_n^2 = 1$ for every $m = 0, 1, \dots$ and $j = 1, \dots, 2^m$, and let $\theta_{m,j}$ be sign numbers ± 1. A *Gaussian Haar system* $(g_{m,j})_{m,j}$ is defined by setting $g_{0,0} = \mathbf{1}_{\widetilde{F}_{0,1}}$ and

$$g_{m,j} = \sum_{n\in\widetilde{N}_{m,j}} \widetilde{\beta}_n \sum_{\operatorname{supp} h_{n,i} \subseteq \widetilde{F}_{m,j}} \theta_{n,i} h_{n,i}$$

for $m = 0, 1, \dots$ and $j = 1, \dots, 2^m$.

Observe that our system $(\widetilde{h}_{m,j})_{m,j}$ is a Gaussian Haar system, by construction. The fact that it is equivalent to the Haar system $(h_{m,j})_{m,j}$ follows directly from [49, Lemma 6.2], which we state here for readers' convenience.

Lemma 7.74. *([49, Lemma 6.2]) Let X be an r.i. function space on* $[0, 1]$ *which is s-concave for some $s < \infty$. If the Haar system $(h_{m,j})_{m,j}$ is unconditional in X then any Gaussian Haar system $(\widetilde{h}_{m,j})_{m,j}$ is also unconditional and equivalent to the Haar system $(h_{m,j})_{m,j}$.*

A proof of Theorem 7.62. Part 2. Proofs of all statements

We now give proofs of all above lemmas and propositions.

Proof of Lemma 7.67. We need the following lemma, the proof of which uses the idea of precise reproducibility of the Haar system.

Lemma 7.75. *Let X be an r.i. function space, $T \in \mathcal{L}(X)$, $(\varepsilon_{0,0}) \cup (\varepsilon_{n,i})_{i=1,n=0}^{2^n\ \infty}$ be any sequence of positive numbers. Then there exists a sequence $(h'_{n,i})$, isometrically equivalent to the Haar system in L_p, and a block basis $(g_{n,i})$ of the Haar system such that $\|Th'_{n,i} - g_{n,i}\| < \varepsilon_{n,i}$ for all indices n, i.*

Proof of Lemma 7.75. We denote by P_m the basic projection of the Haar system onto the linear span of $(h_{0,0}) \cup (h_{n,i})_{i=1,n=0}^{2^n\ m}$, $m \geq 0$. We set $h'_{0,0} = h_{0,0}$. Then choose $m_{0,0}$ so that $\|P_{m_{0,0}} Th'_{0,0} - Th'_{0,0}\| < \varepsilon_{0,0}$ and set $g_{0,0} = P_{m_{0,0}} Th'_{0,0}$.

Let (r_j) be the Rademacher system. Since (Tr_j) is weakly null and $P_{m_{0,0}}$ is a finite rank operator, we can choose $n_{0,1} > m_{0,0}$ so that

$$\left\| P_{m_{0,0}} T r_{n_{0,1}} \right\| < \frac{\varepsilon_{0,1}}{2}. \tag{7.88}$$

Set $h'_{0,1} = r_{n_{0,1}}$. Then choose $m_{0,1} > n_{0,1}$ so that

$$\left\| P_{m_{0,1}} Th'_{0,1} - Th'_{0,1} \right\| < \frac{\varepsilon_{0,1}}{2}. \tag{7.89}$$

Then, putting $g_{0,1} = (P_{m_{0,1}} - P_{m_{0,0}})Th'_{0,1}$, we obtain by (7.88) and (7.89)

$$\|Th'_{0,1} - g_{0,1}\| = \left\| Th'_{0,1} - P_{m_{0,1}} Th'_{0,1} + P_{m_{0,0}} Th'_{0,1} \right\| < \frac{\varepsilon_{0,1}}{2} + \frac{\varepsilon_{0,1}}{2} = \varepsilon_{0,1}.$$

For the next two steps, denote $A_{1,1} = \{t \in [0,1] : h'_{0,1}(t) = 1\}$ and $A_{1,2} = [0,1] \setminus A_{1,1}$. Consider a Rademacher system $(r_j(A_{1,1}))$ in $L_p(A_{1,1})$ (actually, any weakly null sequence of signs supported on $A_{1,1}$). Then choose $n_{1,1} > m_{0,1}$ so that

$$\left\| P_{m_{0,1}} T r_{n_{1,1}}(A_{1,1}) \right\| < \frac{\varepsilon_{1,1}}{2} . \tag{7.90}$$

Set $h'_{1,1} = r_{n_{1,1}}(A_{1,1})$. Then choose $m_{1,1} > n_{1,1}$ so that

$$\left\| P_{m_{1,1}} T h'_{1,1} - T h'_{1,1} \right\| < \frac{\varepsilon_{1,1}}{2} . \tag{7.91}$$

Then, putting $g_{1,1} = (P_{m_{1,1}} - P_{m_{0,1}}) T h'_{1,1}$, we obtain, using (7.90) and (7.91) as above, that $\|T h'_{1,1} - g_{1,1}\| < \varepsilon_{1,1}$.

Analogously, we do the fourth step using a Rademacher system supported on $A_{1,2}$, and choosing $h'_{1,2}$ and $g_{1,2}$. Continuing the construction in this manner, we obtain the desired sequences. □

For the proof of Lemma 7.67 we choose any $\varepsilon_{n,i} > 0$ with $\sum_{(n,i)} 2^{n/p} \varepsilon_{n,i} < 1/2$ (the coefficients $2^{n/p}$ appeared to normalize the Haar system) and, using Lemma 7.75, choose the corresponding sequences $(h'_{n,i})$ and $(g_{n,i})$. By the Krein–Milman–Rutman theorem on stability of basic sequences [79, p. 5], there exists an isomorphism $S : [T h'_{n,i}] \to [g_{n,i}]$ extending the equality $S T h'_{n,i} = g_{n,i}$ for each n, i. Since the sequence $(h'_{n,i})$ is isometrically equivalent to the Haar system, there exists an isometric isomorphism $U : [h_{n,i}] \to [h'_{n,i}]$ extending the equality $U h_{n,i} = h'_{n,i}$ for each n, i. Observe that Uh is a sign whenever h is.

We set $T_1 = STU$. Since $T_1 h_{n,i} = g_{n,i}$ for each n, i, we have that $(T h_{n,i})$ is a block basis of the Haar system, and hence is disjointly supported with respect to the Haar system. Observe that T_1 is sign-embedding. Indeed, if h is a sign then

$$\|T_1 h\| = \|STUh\| \ge \|S^{-1}\|^{-1} \|TUh\| \ge \|S^{-1}\|^{-1} \delta \|Uh\| = \|S^{-1}\|^{-1} \delta \|h\| ,$$

where $\delta > 0$ is taken from the condition $\|Tx\| \ge \delta \|x\|$ which is true for every sign x.

Assume that we have proved the theorem for T_1. Let E be a subspace isomorphic to L_p such that the restriction $T_1|_E$ is an isomorphic embedding. We claim that the restriction $T|_{U(E)}$ is an isomorphic embedding. Indeed, given any $x \in U(E)$, say, $x = Uy$ with $y \in E$, we obtain

$$\begin{aligned} \|Tx\| = \|TUy\| &\ge \|S\|^{-1} \|STUy\| = \|S\|^{-1} \|T_1 y\| \ge \|S\|^{-1} \left\| T_1|_E^{-1} \right\|^{-1} \|y\| \\ &\ge \|S\|^{-1} \left\| T_1|_E^{-1} \right\|^{-1} \|U\|^{-1} \|x\| . \end{aligned}$$

□

Proof of Lemma 7.68. To prove property (P1), we see that, by Corollary 7.66,

$$\begin{aligned}\|v_n\| &\geq \frac{1}{B_p K_p}\Big\|\sum_{i=1}^{2^n} T h_{n,i}\Big\| = \frac{1}{B_p K_p}\Big\|T\Big(\sum_{i=1}^{2^n} h_{n,i}\Big)\Big\| \\ &\geq \frac{\delta}{B_p K_p}\Big\|\sum_{i=1}^{2^n} h_{n,i}\Big\| = \frac{\delta}{B_p K_p} = \gamma > 0\,.\end{aligned}$$

For the proof of property (P2), assume the contrary and choose $\varepsilon_0 > 0$, a subsequence $(v_{n_k})_{k=1}^{\infty}$ and disjoint sets $(A_k)_{k=1}^{\infty}$ so that

$$\int_{A_k} v_{n_k}^p \,\mathrm{d}\mu \geq \varepsilon_0^p$$

for each $k \in \mathbb{N}$. Then for every $m \in \mathbb{N}$

$$\begin{aligned} m^{1/2} &= \Big(\sum_{k=1}^{m} 1^2\Big)^{1/2} = \Big\|\sum_{k=1}^{m}\sum_{i=1}^{2^{n_k}} h_{n_k,i}\Big\|_2 \geq \Big\|\sum_{k=1}^{m}\sum_{i=1}^{2^{n_k}} h_{n_k,i}\Big\|_p \\ &\geq \|T\|^{-1}\Big\|\sum_{k=1}^{m}\sum_{i=1}^{2^{n_k}} T h_{n_k,i}\Big\|_p \\ &\overset{\text{by Cor. 7.66}}{\geq} \|T\|^{-1} A_1 K_p^{-1}\Big\|S\Big(\sum_{k=1}^{m}\sum_{i=1}^{2^{n_k}} T h_{n_k,i}\Big)\Big\|_p \\ &= \|T\|^{-1} A_1 K_p^{-1}\Big\|S^2\Big(\sum_{k=1}^{m}\sum_{i=1}^{2^{n_k}} T h_{n_k,i}\Big)\Big\|_{p/2}^{1/2} \\ &\overset{\text{by (7.74)}}{=} \|T\|^{-1} A_1 K_p^{-1}\Big\|\sum_{k=1}^{m} S^2\Big(\sum_{i=1}^{2^{n_k}} T h_{n_k,i}\Big)\Big\|_{p/2}^{1/2} \\ &= \|T\|^{-1} A_1 K_p^{-1}\Big\|\sum_{k=1}^{m} v_{n_k}^2\Big\|_{p/2}^{1/2} \\ &= \|T\|^{-1} A_1 K_p^{-1}\Big(\int_{[0,1]}\Big|\sum_{k=1}^{m} v_{n_k}^2\Big|^{p/2}\mathrm{d}\mu\Big)^{1/p} \\ &\geq \|T\|^{-1} A_1 K_p^{-1}\Big(\sum_{k=1}^{m}\int_{A_k} v_{n_k}^p\,\mathrm{d}\mu\Big)^{1/p} \\ &\geq \|T\|^{-1} A_1 K_p^{-1}\varepsilon_0 m^{1/p}\,,\end{aligned}$$

which is impossible for large enough m, since $p < 2$.

Finally, note that (P3) follows directly from (P1) and (P2). □

Proof of Lemma 7.69. To prove (P4), we see that (7.79) implies that

$$\int_{[0,1]} \nu\big([0,1]\big)\, \mathrm{d}\mu = \lim_{j\to\infty} \sum_{n\in N_j} \alpha_n \int_{[0,1]} \nu_n\big([0,1]\big)\, \mathrm{d}\mu \,. \tag{7.92}$$

Since $\sum_{n\in N_j} \alpha_n = 1$, it is enough to show that the sequence $\int_{[0,1]} \nu_n([0,1])\, \mathrm{d}\mu$, $n = 0, 1, \ldots$, is uniformly bounded away from zero. By (P3) and the definition of ν_n (7.77) we have

$$\int_{[0,1]} \nu_n\big([0,1]\big)\, \mathrm{d}\mu \geq \Big(\int_{[0,1]} \nu_n\big([0,1]\big)^{p/2}\, \mathrm{d}\mu\Big)^{2/p} \geq \eta^{2/p} \,.$$

Thus, by (7.92),

$$\int_{[0,1]} \nu\big([0,1]\big)\, \mathrm{d}\mu \geq \eta^{2/p} \,, \tag{7.93}$$

which ends the proof of (P4).

To prove (P5), we first observe that since $p/2 < 1$ and $\nu([0,1]) \leq R^2$, (7.93) implies that

$$\int_{[0,1]} \nu([0,1])^{p/2}\, \mathrm{d}\mu \geq \int_{[0,1]} \nu([0,1])\, R^{2(p/2-1)}\, \mathrm{d}\mu \geq \eta^{2/p} R^{p-2} \,. \tag{7.94}$$

On the other hand, for any $A \in \mathcal{E}$ we have the following estimate from above

$$\begin{aligned} \int_{[0,1]} \nu(A)^{p/2}\, \mathrm{d}\mu &= \lim_{j\to\infty} \int_{[0,1]} \Big(\sum_{n\in N_j} \alpha_n \nu_n(A)\Big)^{p/2} \mathrm{d}\mu \\ &\leq \limsup_{j\to\infty} \int_{[0,1]} S^p\Big(T\Big(\sum_{n\in N_j} \alpha_n^{1/2} \sum_{\operatorname{supp} h_{n,i} \subseteq A} h_{n,i}\Big)\Big)\, \mathrm{d}\mu \\ &\leq \big(K_p^2 A_p^{-1} \|T\|\big)^p \mu(A). \end{aligned} \tag{7.95}$$

Next we claim that there exists $\varepsilon > 0$ such that

$$\int_{[0,1]} \max_{1\leq i\leq 2^n} \nu(I_{n,i})^{p/2}\, \mathrm{d}\mu \geq (2\varepsilon)^{p/2}, \quad n = 0, 1, \ldots \,. \tag{7.96}$$

Indeed, by (7.94) and Hölder's inequality for conjugate indices $2/p$ and $2/(2-p)$, we get for any $n = 0, 1, \ldots,$

$$\begin{aligned}
R^{p-2}\eta^{2/p} &\le \int_{[0,1]} \nu\big([0,1]\big)^{p/2}\,\mathrm{d}\mu = \int_{[0,1]} \Big(\sum_{j=1}^{2^n} \nu(I_{n,j})\Big)^{p/2}\,\mathrm{d}\mu \\
&= \int_{[0,1]} \Big(\sum_{j=1}^{2^n} \nu(I_{n,j})^{p/2}\Big)^{p/2} \nu(I_{n,j})^{(1-p/2)(p/2)}\,d\mu \\
&\le \int_{[0,1]} \Big(\sum_{j=1}^{2^n} \nu(I_{n,j})^{p/2}\Big)^{p/2} \max_{1\le i\le 2^n} \nu(I_{n,i})^{(1-p/2)(p/2)}\,d\mu \\
&\le \Big(\int_{[0,1]} \sum_{j=1}^{2^n} \nu(I_{n,j})^{p/2}\,\mathrm{d}\mu\Big)^{p/2} \Big(\int_{[0,1]} \max_{1\le i\le 2^n} \nu(I_{n,i})^{p/2}\,\mathrm{d}\mu\Big)^{(2-p)/2} \\
&\overset{\text{by (7.95)}}{\le} \big(K_p^2 A_p^{-1}\|T\|\big)^{p^2/2} \Big(\int_{[0,1]} \max_{1\le i\le 2^n} \nu(I_{n,i})^{p/2}\,\mathrm{d}\mu\Big)^{(2-p)/2}.
\end{aligned}$$

Thus, the existence of $\varepsilon > 0$ such that (7.96) holds, is proved. Then (7.96) implies

$$\int_{[0,1]} \max_{1\le i\le 2^n} \nu(I_{n,i})\,\mathrm{d}\mu \ge \Big(\int_{[0,1]} \max_{1\le i\le 2^n} \nu(I_{n,i})^{p/2}\,\mathrm{d}\mu\Big)^{2/p} \ge 2\varepsilon\,,$$

Since $\nu(A)$ are uniformly bounded (see (7.80)), there exists $\sigma > 0$ so that for every measurable subset Ω_0 of $[0,1]$ the inequality $\mu(\Omega_0) > 1-\sigma$ implies

$$\int_{\Omega_0} \max_{1\le i\le 2^n} \nu(I_{n,i})\,\mathrm{d}\mu \ge \varepsilon\,,$$

for each $n = 0, 1, \ldots$.

We enumerate $\mathcal{E} = (F_j)_{j=1}^\infty$, and choose a sequence $(\sigma_j)_{j=1}^\infty$ of positive numbers so that $\prod_{j=1}^\infty (1-\sigma_j) > 1-\sigma$. By Egorov's theorem, there exists a measurable subset Ω_j of $[0,1]$ so that the convergence in (7.79) for F_j is uniform on Ω_j. Then for $\Omega' = \bigcap_{j=1}^\infty \Omega_j$ we have that convergence in (7.79) is uniform on Ω' for all $F \in \mathcal{E}$ and, since $\mu(\Omega') > 1-\sigma$, we also have for all $n = 0, 1, \ldots,$

$$\int_{\Omega'} \max_{1\le i\le 2^n} \nu(I_{n,i})\,\mathrm{d}\mu \ge \varepsilon\,,$$

which ends the proof of (P5).

To prove (P6), by (7.78) and (7.95), we obtain

$$\begin{aligned}
\int_{[0,1]} \nu(A)\,\mathrm{d}\mu &= \int_{[0,1]} \nu(A)^{p/2}\cdot \nu(A)^{1-p/2}\,\mathrm{d}\mu \\
&\le R^{2(1-p/2)} \int_{[0,1]} \nu(A)^{p/2}\,\mathrm{d}\mu \\
&\le R^{2-p}\big(K_p^2 A_p^{-1}\|T\|\big)^p \mu(A)\,,
\end{aligned}$$

as required. □

Proof of Lemma 7.70. For each $n = 0, 1, \dots$ and $t \in \Omega'$ we denote

$$M_n(t) = \max_{1 \le i \le 2^n} \nu(I_{n,i})(t) .$$

The sequence $(M_n(t))_{n=0}^{\infty}$ is decreasing for each $t \in \Omega'$. Indeed, let $M_{n+1}(t) = \nu(I_{n+1,j})(t)$, and let i be such that either $j = 2i - 1$ or $j = 2i$. Then

$$M_n(t) \ge \nu(I_{n,i})(t) = \nu(I_{n+1,2i-1})(t) + \nu(I_{n+1,2i})(t) \ge M_{n+1}(t) .$$

Thus, there exists the limit

$$M(t) = \lim_{n \to \infty} M_n(t), \quad t \in \Omega' .$$

By condition (ii) of the assumptions, $\int_{\Omega'} M \, \mathrm{d}\lambda \ge \varepsilon$. Thus there exist $E \in \mathcal{F}(\Omega')$ and $\xi > 0$, such that

$$\int_E M \, \mathrm{d}\lambda \ge \frac{\varepsilon}{2} \tag{7.97}$$

and

$$M(t) \ge \xi \quad \text{for all } \; t \in E . \tag{7.98}$$

We define a sequence of functions $\varphi_n : E \to [0, 1]$ by $\varphi_n(t) = \frac{i-1}{2^n}$, where

$$i = \min\big\{ j \in \{1, \dots, 2^n\} : \nu(I_{n,j})(t) \ge M(t) \big\} .$$

Observe that $(\varphi_n(t))_{n=0}^{\infty}$ is an increasing sequence for each $t \in E$. Indeed, assume $\varphi_n(t) = \frac{j-1}{2^n}$. Then for each $i \le j - 1$ we have

$$M(t) > \nu(I_{n,i})(t) = \nu(I_{n+1,2i-1})(t) + \nu(I_{n+1,2i})(t)$$

Hence, $M(t) > \nu(I_{n+1,s})(t)$ for each $s \le 2j - 2$. This implies that

$$\varphi_{n+1}(t) \ge \frac{2j - 1 - 1}{2^{n+1}} = \varphi_n(t) .$$

Since $(\varphi_n(t))_{n=0}^{\infty}$ is an increasing sequence bounded from above by 1, there exists a limit

$$\varphi(t) = \lim_{n \to \infty} \varphi_n(t)$$

for each $t \in E$. We are going to show that

(a) $\mathbf{1}_{\varphi^{-1}(A)}(t) M(t) \le \nu(A)(t)$ *for every* $A \in \Sigma$ *and* λ*-almost all* $t \in \Omega'$;

(b) *the measure* $\lambda \circ \varphi^{-1}$ *is absolutely continuous with respect to* μ *on* Σ.

Notice that if $A = I_{n,i}$ and $t \in \varphi^{-1}(A)$ then $\varphi(t) \in A$ and therefore, $\varphi_k(t) \in A$ and $\varphi_k(t) + \frac{1}{2^k} \in A$ for large enough k. Thus, by the definition of φ_k ,

$$\nu(A)(t) \geq \nu\big(\varphi_k(t), \varphi_k(t) + \frac{1}{2^k}\,\big)(t) \geq M(t)\,.$$

Hence, in order to complete the proof of (a), it is enough to prove (b).

Let $A \in \Sigma$ and $(I_n)_{n=1}^{\infty}$ be a sequence of disjoint open intervals from [0, 1] such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$. If $t \in \varphi^{-1}(A)$ then $\varphi(t) \in I_n$ for some $n \in \mathbb{N}$, and hence,

$$\nu\Big(\bigcup_{n=1}^{\infty} I_n\Big)(t) \geq \nu(I_n)(t) \geq M(t)\,.$$

Thus,

$$\int_{\varphi^{-1}(A)} \nu(A)\,\mathrm{d}\lambda \geq \int_{\varphi^{-1}(A)} M\,\mathrm{d}\lambda \geq \xi\lambda\big(\varphi^{-1}(A)\big)\,.$$

Then (i) implies that

$$\lim_{\mu(A)\to 0} \lambda\big(\varphi^{-1}(A)\big) = 0\,,$$

which proves (b).

Next we consider the vector measure

$$\mathbf{m}(A) = \Big(\mu(A), \lambda\big(\varphi^{-1}(A)\big), \int_{\varphi^{-1}(A)} M\,\mathrm{d}\lambda\Big)\,.$$

The measure $\mathbf{m}$ is atomless since it is absolutely continuous with respect to μ. Thus, by Lyapunov's theorem, there is a partition $[0,1] = \widetilde{F}_{1,1} \sqcup \widetilde{F}_{1,2}$ into measurable sets with $\mathbf{m}(\widetilde{F}_{1,1}) = \mathbf{m}(\widetilde{F}_{1,2})$. By a suitable small perturbation of these sets, we get a partition $[0,1] = F_{1,1} \sqcup F_{1,2}$ with $F_{1,1}, F_{1,2} \in \mathcal{E}$ such that

$$\mu(F_{1,1}) = \mu(F_{1,2})\,,$$

$$\Big(1 - \frac{1}{2}\Big)\frac{\lambda(E)}{2} \leq \lambda\big(\varphi^{-1}(F_{1,i})\big) \leq \Big(1 + \frac{1}{2}\Big)\frac{\lambda(E)}{2}$$

and

$$\Big(1 - \frac{1}{2}\Big)\frac{1}{2}\int_E M\,\mathrm{d}\lambda \leq \int_{\varphi^{-1}(F_{1,i})} M\,\mathrm{d}\lambda \leq \Big(1 + \frac{1}{2}\Big)\frac{1}{2}\int_E M\,\mathrm{d}\lambda$$

for $i = 1, 2$.

We denote $P_1 = \prod_{j=1}^{\infty}(1 - \frac{1}{2^j})$ and $P_2 = \prod_{j=1}^{\infty}(1 + \frac{1}{2^j})$. By partitioning the sets $F_{1,1}$ and $F_{1,2}$ into subsets from $\mathcal{E}$ in a suitable manner and continuing the process of partitioning to infinity, we obtain a tree $(F_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ with $F_{n,i} \in \mathcal{E}$ such that for all $n = 0, 1, \ldots$ and $i = 1, \ldots, 2^n$

$$P_1\frac{\lambda(E)}{2^n} < \prod_{j=1}^{n+1}\Big(1 - \frac{1}{2^j}\Big)\frac{\lambda(E)}{2^n} \leq \lambda\big(\varphi^{-1}(F_{n,i})\big) \leq \prod_{j=1}^{n+1}\Big(1 + \frac{1}{2^j}\Big)\frac{\lambda(E)}{2^n} < P_2\frac{\lambda(E)}{2^n}$$

and analogously,

$$P_1 \frac{1}{2^n} \int_E M \, \mathrm{d}\lambda < \int_{\varphi^{-1}(F_{m,i})} M \, \mathrm{d}\lambda < P_2 \frac{1}{2^n} \int_E M \, \mathrm{d}\lambda \, .$$

We define $G_{n,i} = \varphi^{-1}(F_{m,i})$ for $n = 0, 1, \ldots$ and $i = 1, \ldots, 2^n$. Clearly, $G_{n,i} \subseteq E \subseteq \Omega'$. We use (a) to finish the proof of Lemma 7.70. □

Proof of Lemma 7.71. We start with $(m, j) = (0, 1)$. Using (P5), we choose $\ell_{0,1} \in \mathbb{N}$ so that

$$\Big| \sum_{n \in N_{\ell_{0,1}}} \alpha_n \nu_n \big(F_{0,1}\big)(t) - \nu\big(F_{0,1}\big)(t) \Big| \le \frac{\xi}{2} \, ,$$

for all $t \in \Omega'$, where $\alpha_n = \alpha_n(F_{0,1})$ and $N_{\ell_{0,1}} = N_{\ell_{0,1}}(F_{0,1})$. We set $N_{0,1} = N_{\ell_{0,1}}(F_{0,1})$ and $\beta_n = \alpha_n(F_{0,1})$ for $n \in N_{0,1}$. Observe that by (7.82), for all $t \in G_{0,1}$,

$$\begin{aligned} \sum_{n \in N_{0,1}} \beta_n \nu_n (F_{0,1})(t) &\ge \nu\big(F_{0,1}\big)(t) - \Big| \sum_{n \in N_{\ell_{0,1}}} \alpha_n \nu_n \big(F_{0,1}\big)(t) - \nu\big(F_{0,1}\big)(t) \Big| \\ &\ge \xi - \frac{\xi}{2} = \frac{\xi}{2} \, . \end{aligned}$$

Suppose that for all (m, j) with $2^m + j < 2^{m_0} + j_0$, sets $N_{m,j}$ and numbers $(\beta_n)_{n \in N_{m,j}}$ are constructed. Now we will construct N_{m_0,j_0} and β_n for $n \in N_{m_0,j_0}$. Using (P5), we choose $\ell_{m_0,j_0} \in \mathbb{N}$ so that

$$\Big| \sum_{n \in N_{\ell_{m_0,j_0}}} \alpha_n \nu_n \big(F_{m_0,j_0}\big)(t) - \nu\big(F_{m_0,j_0}\big)(t) \Big| \le \frac{\xi}{2}$$

for all $t \in \Omega'$, where $\alpha_n = \alpha_n(F_{m_0,j_0})$ and $N_{\ell_{m_0,j_0}} = N_{\ell_{m_0,j_0}}(F_{m_0,j_0})$, and the sets $N_{m_0,j_0} = N_{\ell_{m_0,j_0}}(F_{m_0,j_0})$ satisfy the additional properties:

(P7) $\min N_{m_0,j_0} > \max N_{m,j}$ for each pair (m, j) such that $2^m + j < 2^{m_0} + j_0$;

(P8) $\min \widetilde{N}_{m_0,j_0} > \min\{\ell : \widetilde{F}_{m_0,j_0} \in \mathcal{E}_\ell\}$.

(Property (P7) guarantees disjointness, and property (P8) will be used later.)

Then, setting $\beta_n = \alpha_n(F_{m_0,j_0})$ for $n \in N_{m_0,j_0}$, we obtain by (7.82)

$$\begin{aligned} &\sum_{n \in N_{m_0,j_0}} \beta_n \nu_n \big(F_{m_0,j_0}\big)(t) \\ &\qquad \ge \nu\big(F_{m_0,j_0}\big)(t) - \Big| \sum_{n \in N_{\ell_{m_0,j_0}}} \alpha_n \nu_n \big(F_{m_0,j_0}\big)(t) - \nu\big(F_{m_0,j_0}\big)(t) \Big| \\ &\qquad \ge \xi - \frac{\xi}{2} = \frac{\xi}{2} \, . \end{aligned}$$

It remains to notice that the condition $\sum_{n \in N_{m,j}} \beta_n = 1$ follows from the corresponding condition for $\alpha_n(F_{m,j})$. □

Proof of Proposition 7.72. For the proof we need two following statements.

Lemma 7.76 (Stein, Johnson, Maurey, Schechtman, Tzafriri, [49]). *Let X be an r.i. function space on $[0,1]$, whose Boyd indices satisfy $0 < \beta_X \le \alpha_X < 1$, and let $(E_n)_{n=1}^{\infty}$ be a sequence of conditional expectation operators with respect to an increasing sequence of sub-σ-algebras of the Lebesgue σ-algebra on $[0,1]$. Then there exists a constant K_1 so that*

$$\Big\| \Big(\sum_{n=1}^{\infty} (E_n x_n)^2 \Big)^{1/2} \Big\| \le K_1 \Big\| \Big(\sum_{n=1}^{\infty} x_n^2 \Big)^{1/2} \Big\|$$

for any sequence $(x_n)_{n=1}^{\infty}$ in X.

Lemma 7.77 (Johnson, Maurey, Schechtman, Tzafriri, [49]). *Let X be an r.i. function space on $[0,1]$ with $0 < \beta_X \le \alpha_X < 1$. Then there exists a constant $K_2 > 0$ so that*

$$K_2^{-1} \Big\| \sum_{n,i} a_{n,i} h_{n,i} \Big\| \le \Big\| \Big(\sum_{n,i} a_{n,i}^2 h_{n,i}^2 \Big)^{1/2} \Big\| \le K_2 \Big\| \sum_{n,i} a_{n,i} h_{n,i} \Big\|$$

for any sequence $(a_{n,i})$ of scalars.

We remark that Lemma 7.77 for the case when $X = L_p$ is exactly Lemma 7.65 applied to the unconditional sequence $(a_{n,i} h_{n,i})$, and for the general case it is an easy consequence of the above-mentioned Maurey statement [80, p. 50] for a q-concave Banach lattice X for some $q < \infty$.

To prove Proposition 7.72, we will use Lemma 7.77 and Lemma 7.76 for the conditional expectation operators with respect to the finite σ-algebra generated by the sets $(G_{n,i})_{i=1}^{2^n}$ and the functions $x_n = \sum_{i=1}^{2^n} |a_{n,i}| S(k_{n,i}) \mathbf{1}_{G_{n,i}}$, $n = 0, 1, \ldots$.

Let $\{a_{0,0}\} \cup (a_{m,j})_{m=0}^{\infty}\,{}_{j=1}^{2^m}$ be any sequence of scalars. To avoid huge notation, we assume that $a_{0,0} = 0$, however one can see from the proof below that all the inequalities are true for the general case.

Since $(k_{n,i})$ is a block basis of some enumeration of the Haar system, we have that

$$S^2 \Big(\sum_{n=0}^{\infty} \sum_{i=1}^{2^n} a_{n,i} k_{n,i} \Big) = \sum_{n=0}^{\infty} \sum_{i=1}^{2^n} a_{n,i}^2 S^2(k_{n,i}) \,.$$

Another simple observation is that Lemma 7.77 asserts that

$$K_2^{-1} \|x\| \le \|S(x)\| \le K_2 \|x\| \tag{7.99}$$

for every $x \in X$. Thus, we get

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}k_{n,i}\Big\| &\overset{\text{by (7.99)}}{\geq} K_2^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2 S^2(k_{n,i})\Big)^{1/2}\Big\| \qquad (7.100)\\
&\geq K_2^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2 S^2(k_{n,i})\mathbf{1}_{G_{n,i}}\Big)^{1/2}\Big\|\\
&= K_2^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\Big[\sum_{i=1}^{2^n} |a_{n,i}|S(k_{n,i})\mathbf{1}_{G_{n,i}}\Big]^2\Big)^{1/2}\Big\|\\
&\overset{\text{by Lemma 7.76}}{\geq} K_2^{-1}K_1^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\Big[E_n\Big(\sum_{i=1}^{2^n} |a_{n,i}|S(k_{n,i})\mathbf{1}_{G_{n,i}}\Big)\Big]^2\Big)^{1/2}\Big\|\\
&= K_2^{-1}K_1^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\Big[\sum_{i=1}^{2^n} \frac{1}{\mu(G_{n,i})}\int_{G_{n,i}} |a_{n,i}|S(k_{n,i})\,\mathrm{d}\mu\Big]^2\mathbf{1}_{G_{n,i}}\Big)^{1/2}\Big\|.
\end{aligned}$$

By (ii) and (iii) we have that

$$\int_{G_{n,i}} S(k_{n,i})\,\mathrm{d}\mu \geq \frac{1}{C^2 2^n} \quad \text{for } n = 0, 1, \dots \quad \text{and } i = 1, \dots, 2^n .$$

Thus, we can continue to estimate (7.100)

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}k_{n,i}\Big\| &\geq K_2^{-1}K_1^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} \frac{2^{2n}}{C^2}a_{n,i}^2\frac{1}{C^4 2^{2n}}\mathbf{1}_{G_{n,i}}\Big)^{1/2}\Big\| \qquad (7.101)\\
&\geq K_2^{-1}K_1^{-1}C^{-3}\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2\mathbf{1}_{G_{n,i}}\Big)^{1/2}\Big\|.
\end{aligned}$$

Now let $(H_{n,i})_{n=0}^{\infty}{}_{i=1}^{2^n}$ be a sequence of measurable subsets of $[0, 1]$ such that $H_{n,i} = H_{n+1,2i-1} \sqcup H_{n+1,2i}$ and $\mu(H_{n,i}) = C^{-1}2^{-n}$ for $n = 0, 1, \dots$ and $i = 1, \dots, 2^n$. Since $\mu(G_{n,i}) \geq \mu(H_{n,i})$ for all indices n, i, we have

$$\begin{aligned}
\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2\mathbf{1}_{G_{n,i}}\Big)^{1/2}\Big\| &\geq \Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2\mathbf{1}_{H_{n,i}}\Big)^{1/2}\Big\| \qquad (7.102)\\
&= C^{-1}\Big\|\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}^2 h_{n,i}^2\Big)^{1/2}\Big\|\\
&\overset{\text{by Lemma 7.77}}{\geq} C^{-1}K_2^{-1}\Big\|\sum_{n=0}^{\infty}\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\Big\|,
\end{aligned}$$

which together with (7.101) and (i) prove that $(k_{n,i})$ is equivalent to $(h_{n,i})$. □

Thus, proofs of all statements are finished. Theorem 7.62 and therefore, Theorem 7.55 are proved.

7.4 An application to almost isometric copies of L_1

The well-known James theorem [79, Proposition 2.e.3] says that if a Banach space E is isomorphic to ℓ_1 then for each $\varepsilon > 0$ there exists a subspace $E_0 \subseteq E$ which is $(1+\varepsilon)$-isomorphic to ℓ_1. The same is not true for L_1: for each $\lambda > 1$ there exists a Banach space E isomorphic to L_1 which contains no subspace λ-isomorphic to L_1 [78]. However, as an application of Theorem 7.46, we can prove that if a subspace of L_1 is isomorphic to L_1 then it contains an almost isometric copy of L_1.

Theorem 7.78 (Rosenthal [128]). *Let X be a subspace of L_1 isomorphic to L_1. Then for each $\varepsilon > 0$ there exists a subspace Y of X which is $(1+\varepsilon)$-isomorphic to L_1.*

Theorem 7.78 is a consequence of the following more general result which will also be used below.

Theorem 7.79. *Let $S \in \mathcal{L}(L_1)$ be an into isomorphism. Then for each $\varepsilon > 0$ there exist $A, B \in \Sigma^+$, atomless sub-σ-algebras $\Sigma_1 \subseteq \Sigma(A)$ and $\Sigma_2 \subseteq \Sigma(B)$, and an isomorphism $J : L_1(A, \Sigma_1) \to L_1(B, \Sigma_2)$ such that*

(a) *$\widetilde{S} = S|_{L_1(A,\Sigma_1)}$ is an into isomorphism with $\|\widetilde{S}\|\|\widetilde{S}^{-1}\| < 1+\varepsilon$;*

(b) *$\|\widetilde{S} - J\| < \varepsilon$.*

Proof of Theorem 7.79. Fix $\varepsilon > 0$. By Theorem 7.46, $S = S_{pe} + S_n$, where S_{pe} is a pseudo-embedding and S_n is narrow. Since S is not narrow, $S_{pe} \neq 0$. By Theorem 7.39, there exists $A_0 \in \Sigma^+$ so that $S_0 = S_{pe}|_{L_1(A_0)}$ is an into isomorphism with $\|S_0\|\|S_0^{-1}\| < 1+\varepsilon$, and a d.p.o. $U_0 : L_1(A_0) \to L_1$ with $\|S_0 - U_0\| < \varepsilon/2$. Since the operator $S_1 = S_n|_{L_1(A)}$ is narrow, by Theorem 2.21, there exists a Haar-type system (g_n) in $L_1(A_0)$ so that $\|S_1|_E\| \le \delta$, where $E = [g_n]$ and $\delta \in (0, \varepsilon/2)$ is so small that

$$1 \le \frac{\|S_0\| + \delta}{\|S_0^{-1}\|^{-1} - \delta} < 1 + \varepsilon\,. \tag{7.103}$$

Let $\widetilde{A} = \operatorname{supp} g_1 \subseteq A_0$. Observe that $E = L_1(\widetilde{A}, \Sigma_1)$ where Σ_1 is the sub-σ-algebra of $\Sigma(\widetilde{A})$ generated by the sequence (g_n). Since $U = U_0|_E : L_1(\widetilde{A}, \Sigma_1) \to L_1$ is still a d.p.o., we can apply Proposition 7.36 to find $A \in \Sigma_1^+$, $B \in \Sigma^+$ and a sub-σ-algebra Σ_2 of $\Sigma(B)$ such that $J = U|_{L_1(A,\Sigma_1)} : L_1(A, \Sigma_1) \to L_1(B, \Sigma_2)$ is an isomorphism (this implies that Σ_2 is atomless).

If $x \in L_1(A, \Sigma_1) \subseteq E$ then

$$\|Sx\| \le \|S_{pe}x\| + \|S_nx\| = \|S_0x\| + \|S_1x\| \le \big(\|S_0\| + \delta\big)\|x\|$$

and

$$\|Sx\| \geq \|S_0 x\| - \|S_1 x\| \geq \big(\|S_0^{-1}\|^{-1} - \delta\big)\|x\| \,.$$

Hence for $\widetilde{S} = S|_{L_1(A,\Sigma_1)}$ by (7.103) we obtain

$$\|\widetilde{S}\|\|\widetilde{S}^{-1}\| \leq \frac{\|S_0\| + \delta}{\|S_0^{-1}\|^{-1} - \delta} < 1 + \varepsilon \,.$$

It remains to observe that

$$\begin{aligned} \|\widetilde{S} - J\| &\leq \big\|\widetilde{S} - S_0|_{L_1(A,\Sigma_1)}\big\| + \big\|S_0|_{L_1(A,\Sigma_1)} - J\big\| \\ &\leq \big\|S|_E - S_{pe}|_E\big\| + \|S_0 - U_0\| \leq \|S_1|_E\| + \frac{\varepsilon}{2} \leq \delta + \frac{\varepsilon}{2} < \varepsilon \,. \end{aligned}$$

□

Another consequence of Theorem 7.79 is the following result of Rosenthal [128, Theorem 2.1], which will be used in Section 8.5.

Theorem 7.80. *Let X be a Banach space and suppose that $T \in \mathcal{L}(L_1, X)$ fixes a copy of L_1. Then there exists $A_0 \in \Sigma^+$ and an atomless sub-σ-algebra Σ_0 of $\Sigma(A_0)$ such that the restriction $T|_{L_1(A_0,\Sigma_0)}$ is an into isomorphism.*

Proof. Let Y be a subspace of L_1 isomorphic to L_1 such that $T|_Y$ is an into isomorphism, and let $S : L_1 \to Y$ be an isomorphism. Let $\delta > 0$ be such that $\|Ty\| \geq \delta\|y\|$ for each $y \in Y$. For $\varepsilon = \delta\,\|S^{-1}\|^{-1}\|T\|^{-1}/2$ we choose by Theorem 7.79 $A, B \in \Sigma^+$, an atomless sub-σ-algebra Σ_1 of $\Sigma(A)$, an atomless sub-σ-algebra Σ_2 of $\Sigma(B)$ and an isomorphism $J : L_1(A, \Sigma_1) \to L_1(B, \Sigma_2)$ such that (a) and (b) hold. Then for any $x \in L_1(B, \Sigma_2)$ we have

$$\begin{aligned} \|Tx\| &\geq \|TSJ^{-1}x\| - \|T\|\|JJ^{-1}x - SJ^{-1}x\| \\ &\geq \delta\|SJ^{-1}x\| - \|T\|\|J - \widetilde{S}\|\|J^{-1}x\| \\ &\geq \big(\delta\|S^{-1}\|^{-1} - \|T\|\varepsilon\big)\|J\|^{-1}\|x\| = \frac{\delta}{2\|S^{-1}\|\|J\|}\|x\| \,. \end{aligned}$$

Thus, the thesis of the theorem is valid for $A_0 = B$ and $\Sigma_0 = \Sigma_2$. □

It is unknown whether Theorem 7.80 is true for the spaces L_p.

Open problem 7.81. Suppose $1 < p < \infty$, $p \neq 2$. Let X be a Banach space and suppose that $T \in \mathcal{L}(L_p, X)$ fixes a copy of L_p. Does there exist $A_0 \in \Sigma^+$ and an atomless sub-σ-algebra Σ_0 of $\Sigma(A_0)$ such that the restriction $T|_{L_p(A_0,\Sigma_0)}$ is an into isomorphism? What if $X = L_p$?

7.5 An application to complemented subspaces of L_p

Here we show that "well" co-complemented subspaces of L_p are isomorphic to L_p.

Proposition 7.82 (Plichko and Popov [110]). *Let $1 \le p \le 2$ and let $L_p = X \oplus Y$ be a decomposition into subspaces with Y nonisomorphic to L_p. Then X is a rich subspace.*

Proof. We show that the projection Q of L_p onto Y with $\ker Q = X$ is narrow. Since Y is not isomorphic to L_p, by [80, Proposition 2.d.5] for $1 < p < \infty$, Y contains no isomorphic copy of L_p (the same for $p = 1$ follows from [37]), and thus, Q is a non-Enflo operator. Then we consider the following cases to show that Q is narrow.

(a) $p = 1$. The assertion follows from Theorem 7.30.

(b) $1 < p < 2$. The assertion follows from Theorem 7.55.

(c) $p = 2$. In this case the proposition is obvious. □

We do not know whether Proposition 7.82 is true for $2 < p < \infty$. The standard duality argument does not work, since if $L_p = X \oplus Y$ is a decomposition into subspaces with X rich, then is does not follow that $Y^\perp$ is rich in L_q [118] (where $Y^\perp = \{f \in L_q : \forall y \in Y,\ \langle f, y\rangle = 0\}$). Indeed, by Corollary 4.16, there are two decompositions $L_p = E_0 \oplus E_1 = E_0 \oplus E_2$ with E_1 rich and E_2 not rich. If $Y^\perp$ was always rich, then since E_1 is rich, $E_0^\perp$ would also be rich, and so would be E_2, which is false. However, this does not provide a counterexample to Proposition 7.82 for $2 < p < \infty$, because all the spaces E_0, E_1, E_2 are isomorphic to L_p.

We have the following consequence of Corollary 6.12 and Proposition 7.82. Let k_p be the best constant for which Corollary 6.12 is true.

Corollary 7.83. *Let $1 \le p < \infty$ and let $P : L_p \to X$ be a projection onto a subspace X so that*

- $\|I - P\| < k_p$, *if* $1 \le p \le 2$;
- $\|I - P\| < k_q$, *where* $q = p/(p-1)$, *if* $2 < p < \infty$.

Then X is isomorphic to L_p.

Proof. The proof in the case $1 \le p \le 2$ is immediate. Assume $2 < p < \infty$. Then $L_q = X^\perp \oplus (\ker P)^\perp$. Moreover, $(I - P)^*$ is the projection of L_q onto $(\ker P)^\perp$ along $X^\perp$. Suppose on the contrary that X is not isomorphic to L_p. Since the conjugate operator to an isomorphism is an isomorphism, $X^\perp$ is not isomorphic to L_q. By Proposition 7.82, $(\ker P)^\perp$ is rich, and by Corollary 6.12, $\|I - P\| = \|(I - P)^*\| \ge k_q$, which is a contradiction. □

To the best of our knowledge, the following closely related Alspach's problem (see [7]) is still unsolved.

Let $1 \le p < \infty$, $p \ne 2$. Does there exist a constant $\lambda_p > 1$ such that every λ_p-complemented subspace of L_p is isomorphic to L_p?

(Recall that every 1-complemented subspace of L_p is isometric to an L_p-space.)

7.6 Narrow operators defined on rich subspaces, and the Daugavet property for rich subspaces of L_1

In this section we define the notion of a narrow operator on a rich subspace X of a Köthe–Banach space. We then show that any rich subspace X of $L_1(\mu)$ with a finite atomless measure μ has the Daugavet property with respect to narrow operators on X and L_1-strictly singular operators. The results are of a very similar nature, but for $C[0,1]$, the results are presented in Section 11.3.

The results of this section were obtained by V. Kadets and Popov in [57].

Definition 7.84. Let E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ), X a rich subspace of E and Y, a Banach space. An operator $T \in \mathcal{L}(X, Y)$ is called *narrow on* X if for each $A \in \Sigma$ and each $\varepsilon > 0$ there exist a mean zero sign r on A and $x \in X$ such that $\|x - r\| < \varepsilon$ and $\|Tx\| < \varepsilon$.

We remark that for general spaces the restriction of a narrow operator to a rich subspace X need not be narrow on X.

Example 7.85. Suppose a Köthe–Banach space E is represented as a direct sum of rich subspaces $E = X \oplus Y$ (for example, this is the case if E is an r.i. space with an unconditional basis, see Corollary 5.3). Then the projection P of E onto X parallel to Y is a narrow operator, and the restriction $P|_X$, being the identity of X, is not narrow on X.

We do not know whether the restriction of a narrow operator on L_1 to a rich subspace X of L_1 is always narrow on X, however we do have the following result.

Theorem 7.86. *Let X be a rich subspace of $L_1(\mu)$. Then every L_1-strictly singular operator $T \in \mathcal{L}(X)$ is narrow on X.*

For the proof we need some lemmas.

Lemma 7.87. *Let X be a rich subspace of $L_1(\mu)$, $A \in \Sigma^+$ and $\varepsilon \in (0, 1/2)$. Then there are $x_{n,k} \in X$ and $A_{n,k} \in \Sigma^+$, $n \in \mathbb{N}$, $k = 1, \dots, 2^{n-1}$ such that*

(i) $A_{1,1} = A, \;\; A_{n,k} = A_{n+1,2k-1} \sqcup A_{n+1,2k}, \;\; \mu(A_{n,k}) = \dfrac{\mu(A)}{2^{n-1}};$

(ii) $\|x_{n,k} - h_{n,k}\| < \dfrac{\varepsilon}{8^n}, \quad$ *where* $\quad h_{n,k} = \mathbf{1}_{A_{n+1,2k-1}} - \mathbf{1}_{A_{n+1,2k}}.$

Proof of Lemma 7.87. Since the set $A_{1,1}$ is given, it is sufficient to show how to construct the sets $A_{n+1,2k-1}$, $A_{n+1,2k}$ and the function $x_{n,k}$, once a set $A_{n,k}$ is known. By Definition 1.7 there exist $x \in X$ and a mean zero sign $y \in L_1(\mu)$ on the set $A_{n,k}$ such that $\|x - y\| < \varepsilon/8^n$. Let $x_{n,k} = x$, and

$$A_{n+1,2k-1} = \{t \in A_{n,k} : y(t) = 1\}, \quad A_{n+1,2k} = \left\{t \in A_{n,k} : y(t) = -1\right\}.$$

Then conditions (i) and (ii) hold. □

Lemma 7.88. *Let X be a rich subspace of $L_1(\mu)$, $A \in \Sigma^+$ and $\theta \in (0, 1/2)$. Then there exists an atomless sub-σ-algebra Σ_1 of $\Sigma(A)$, a subspace Y of X isomorphic to L_1 and complemented in $L_1(\mu)$, and a projection P from $L_1(\mu)$ onto Y such that for every $y \in L_1^0(A, \Sigma_1, \mu) \stackrel{\text{def}}{=} \{x \in L_1(A, \Sigma_1, \mu) : \int_A x \, d\mu = 0\}$ we have*

$$\|Py - y\| \le \theta \|y\|. \tag{7.104}$$

Proof of Lemma 7.88. Let $\varepsilon = \theta/100$, and $A_{n,k}$, $x_{n,k}$ and $h_{n,k}$ be as in Lemma 7.87. Let Σ_1 be the least σ-algebra containing all $A_{n,k}$. Then $L_1(A, \Sigma_1, \mu)$ is isometrically isomorphic to L_1 (a natural isometry is obtained by extending the equality $Jh_{n,k} = \mu(A) \cdot \tilde{h}_{2^{n-1}+k}$, where $(\tilde{h}_i)$ is the Haar system in L_1). Thus the system $(h_{n,k} : n \in \mathbb{N}, k = 1, \dots, 2^{n-1})$ is a basis of the subspace $Z = L_1^0(A, \Sigma_1, \mu)$, which is a subspace of codimension one in $L_1(A, \Sigma_1, \mu)$, and hence, is isomorphic to L_1. By the Krein–Milman–Rutman theorem on the stability of basic sequences ([52, p. 64], [79, p. 5]), the system $(x_{n,k})$ in its natural order is a basis of its closed linear span Y. Moreover, the operator $T \in \mathcal{L}(Z, Y)$, which extends the equalities $Th_{n,k} = x_{n,k}$, is an isomorphism and

$$\max\left\{\|T\|, \|T^{-1}\|\right\} \le 1 + \varepsilon, \quad \|z - Tz\| \le \varepsilon \|z\| \tag{7.105}$$

for all $z \in Z$.

Thus, Y is a subspace of X isomorphic to L_1 and complemented in L_1.

Now we will construct a projection P. Denote by $V_1 : L_1(\mu) \to L_1(A)$ the restriction operator $V_1 x = x \cdot \mathbf{1}_A$ for each $x \in L_1(\mu)$, by $V_2 : L_1(A) \to L_1(A, \Sigma_1, \mu)$ the conditional expectation operator with respect to Σ, by $V_3 : L_1(A, \Sigma_1, \mu) \to Z$ the projection, defined by

$$V_3 x = x - \frac{1}{\mu(A)} \int_A x \, d\mu, \quad x \in L_1(A, \Sigma_1, \mu).$$

By [80, p. 122], $\|V_2\| = 1$. It is also not hard to show that $\|V_1\| = 1$ and $\|V_3\| = 2$. Therefore, for the projection $V = V_3 \circ V_2 \circ V_1$ from L_1 onto Z we have $\|V\| \le 2$. Let $U = T \circ V \in \mathcal{L}(L_1, Y)$, $W = U|_Y \in \mathcal{L}(Y)$ and $I_Y \in \mathcal{L}(Y)$ be the identity operator on Y. By (7.105) we get for every $y \in Y$

$$\begin{aligned}
&\|y - T^{-1}y\| = \|T\,(T^{-1}y) - T^{-1}y\| \le \varepsilon\,\|T^{-1}y\| \le 2\varepsilon\,\|y\|, \\
&\|T^{-1}y - Vy\| = \|V\,(T^{-1}y - y)\| \le 4\varepsilon\,\|y\|, \\
&\|Vy - TVy\| \le \varepsilon\,\|Vy\| \le 2\varepsilon\,\|y\|.
\end{aligned}$$

Hence,

$$\|(I_Y - W)\,y\| = \|y - Uy\| = \|y - TVy\| \le \\ \le \|y - T^{-1}y\| + \|T^{-1}y - Vy\| + \|Vy - TVy\| \le 8\varepsilon\,\|y\|\,.$$

Thus, $\|I_Y - W\| \le 8\varepsilon$, and therefore the operator W is invertible with $\|W^{-1}\| \le 1 + 16\varepsilon$.

Set $P = W^{-1}U$. Observe that $P : L_1 \to Y$ and $P|_Y = I_Y$, that is, P is a projection onto Y of norm $\|P\| < 3$. It remains to prove (7.104). Let $z \in Z$. Then

$$\|Pz - z\| \le \|Pz - P(Tz)\| + \|Tz - z\| \le \big(\|P\| + 1\big)\,\varepsilon\,\|z\| < \theta\|z\|\,. \qquad \square$$

Proof of Theorem 7.86. Assume for contradiction, that $T \in \mathcal{L}(X)$ is L_1-strictly singular and not narrow on X. Then there are $A \in \Sigma^+$ and $\varepsilon > 0$ such that for any $x \in X$ and any mean zero sign $y \in L_1(\mu)$ on A, the inequality $\|x - y\| \le \varepsilon$ implies that $\|Tx\| > \varepsilon$. For A and $\theta = \varepsilon$, we construct a projection P and a σ-algebra Σ_1 satisfying properties of Lemma 7.88. Then the operator $\tilde{T} = T \circ P|_{L_1(A,\Sigma_1,\mu)}$ acts from $L_1(A, \Sigma_1, \mu)$ to X. Given a mean zero sign $y \in L_1^0(A, \Sigma_1, \mu)$, by (7.104), we obtain that $\|Py - y\| < \varepsilon\,\|y\| < \varepsilon$, and hence, $\|\tilde{T}y\| = \|T\,(Py)\| > \varepsilon$. Thus, the operator $\tilde{T}$ is not narrow on X.

By Theorem 7.45, there exists $B \in \Sigma_1^+$ such that $\tilde{T}|_{L_1(B,\Sigma_1(B),\mu)}$ is an isomorphic embedding. On the other hand, since $\tilde{T}|_{L_1(B,\Sigma_1(B),\mu} = TP|_{L_1(B,\Sigma_1(B),\mu)}$, we have that $P|_{L_1(B,\Sigma_1(B),\mu)}$ is an isomorphism from $L_1(B, \Sigma_1(B), \mu)$ onto $PL_1(B, \Sigma_1(B), \mu) \subseteq X$, that is, $PL_1(B, \Sigma_1(B), \mu)$ is isomorphic to L_1 and T is bounded from below on $PL_1(B, \Sigma_1(B), \mu)$. In other words, T fixes a copy of L_1, which is a contradiction. $\square$

The following lemma which will be needed for the proof of the Daugavet property for rich subspaces of $L_1(\mu)$, is an analog of Lemma 1.11.

Lemma 7.89. *Let E be a Köthe–Banach space, Y a Banach space, X a rich subspace and $T \in \mathcal{L}(X, Y)$ a narrow operator on X. Then for every $A \in \Sigma$, $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $x \in X$ and a decomposition $A = A' \sqcup A''$ with $A', A'' \in \Sigma$ of measure $\mu(A') = (1-2^{-n})\,\mu(A)$ and $\mu(A'') = 2^{-n}\,\mu(A)$ such that for $h = \mathbf{1}_{A'} - (2^n - 1)\,\mathbf{1}_{A''}$ we have $\|x - h\| < \varepsilon$ and $\|Tx\| < \varepsilon$.*

We omit the proof of Lemma 7.89, which is very similar to the proof of Lemma1.11.

Theorem 7.90. *Every rich subspace X of L_1 has the Daugavet property with respect to the class of narrow operators on X.*

Proof. Let X be a rich subspace of L_1, $T \in \mathcal{L}(X)$ a narrow operator on X, and $\varepsilon > 0$. Let $x \in S_X$ with $\|Tx\| > \|T\| - \varepsilon$. Fix $n \in \mathbb{N}$ so that $\|\mathbf{1}_B \cdot Tx\| < \varepsilon$ for any

$B \in \Sigma^+$ such that $\mu(B) \le 2^{-n}$. Choose a simple function

$$x_0 = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k} \in S_{L_1}$$

with disjoint (A_k) so that $\|x - x_0\| < \varepsilon$. By Lemma 7.89, for every A_k, there exists a decomposition $A_k = A'_k \sqcup A''_k$ with $\mu(A'_k) = (1-2^{-n})\,\mu(A_k)$, $\mu(A''_k) = 2^{-n}\,\mu(A_k)$, and functions $x_k \in X$, $h_k = \mathbf{1}_{A'_k} - (2^n - 1)\mathbf{1}_{A''_k}$, for which $\|x_k - h_k\| < \varepsilon\,\mu(A_k)$ and $\|Tx_k\| < \varepsilon\,\mu(A_k)$. Then for $z = \sum_{k=1}^m a_k x_k$, $h = \sum_{k=1}^m a_k h_k$ we obtain

$$\|Tz\| \le \sum_{k=1}^{m} |a_k|\,\varepsilon\,\mu(A_k) = \varepsilon\,\|x_0\| = \varepsilon\,, \tag{7.106}$$

$$\|z - h\| \le \sum_{k=1}^{m} |a_k|\,\|x_k - h_k\| < \varepsilon\,, \tag{7.107}$$

$$\|x_0 + h\| \le \left\| \sum_{k=1}^{m} a_k 2^n \mathbf{1}_{A''_k} \right\| = \sum_{k=1}^{m} |a_k|\,\mu(A_k) = \|x_0\| = 1\,. \tag{7.108}$$

Observe that the support $B = \bigcup_{k=1}^m A''_k$ of the function $x_0 + h$ is of measure $\mu(B) \le 2^{-n}$, and thus, $\|\mathbf{1}_B \cdot Tx\| < \varepsilon$, by the choice of n. Hence,

$$\begin{aligned} \|(x_0 + h) + Tx\| &\ge \|(x_0 + h) + \mathbf{1}_{[0,1]\setminus B)} \cdot Tx\| - \|\mathbf{1}_B \cdot Tx\| \\ &= \|x_0 + h\| + \left\|\mathbf{1}_{[0,1]\setminus B} \cdot Tx\right\| - \left\|\mathbf{1}_B \cdot Tx\right\| \\ &\ge \|x_0 + h\| + \|Tx\| - 2\left\|\mathbf{1}_B \cdot Tx\right\| \ge \|x_0 + h\| + \|Tx\| - 2\varepsilon \\ &\ge 1 + \|T\| - 3\varepsilon\,. \end{aligned} \tag{7.109}$$

By (7.107) and (7.108), we obtain

$$\|x + z\| \le \|x - x_0\| + \|x_0 + h\| + \|z - h\| \le 1 + 2\varepsilon\,.$$

By (7.106) and (7.109), we obtain

$$\begin{aligned} (1 + 2\varepsilon)\|I + T\| &\ge \|(I + T)(x + z)\| = \|x + z + Tx + Tz\| \\ &\ge \|x_0 + h + Tx\| - \|x_0 - x\| - \|z - r\| - \|Tz\| \\ &\ge 1 + \|T\| - 6\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved. □

Corollary 7.91. *Let Y be a Banach space that embeds into L_1. There exists an equivalent norm $|||\cdot|||$ on the space $X = L_1 \oplus Y$ so that $(X, |||\cdot|||)$ has the Daugavet property with respect to L_1-strictly singular operators.*

Proof. By Corollary 4.16, there exists a decomposition of the space L_1 into a direct sum of subspaces $L_1 = V \oplus W$, both isomorphic to L_1, one of which (say, W) is rich. Let V_1 be a subspace of V isomorphic to Y. Then $V_1 \oplus W$ is a rich subspace of L_1 (because it contains W), isomorphic to X. By Theorem 7.90, the space $V_1 \oplus W$ has the Daugavet property with respect to narrow operators on X, and, in particular, to L_1-strictly singular operators (see Theorem 7.86). Hence, X is isomorphic to a Banach space having the Daugavet property with respect to L_1-strictly singular operators. □

Chapter 8

Weak embeddings of L_1

There are several weakenings of the notion of an isomorphic embedding of Banach spaces, which still preserve some properties of isomorphic embeddings. Here we consider three of them: sign-embeddings, semi-embeddings and G_δ-embeddings. All of these notions have close connections with narrow operators.

This chapter contains an analysis of the relationships between them due to Mykhaylyuk and Popov [101] (2006), and related important results due to Ghoussoub and Rosenthal [44] (1984), Rosenthal [126] (1981) and Talagrand [138] (1990).

8.1 Definitions

Sign-embeddings

The following notion was introduced by Rosenthal in [126] (1981), see also [127].

Definition 8.1. Let X be a Banach space. An injective operator $T \in \mathcal{L}(L_1, X)$ is called a *sign-embedding* if there exists $\delta > 0$ so that for every sign $x \in L_1$,

$$\|Tx\| \geq \delta \|x\| \ . \tag{8.1}$$

Recall, that in Chapter 7 we considered a similar notion of operators satisfying (8.1) but without the assumption of injectivity. We called that notion a generalized sign-embedding (see Definition 7.61). Below we construct an example of a projection of L_1 which is a generalized sign-embedding and with kernel isomorphic to L_1 (Example 8.14). Thus the injectivity assumption is essential in Definition 8.1.

Clearly, the notions of sign-embeddings and narrow operators are mutually exclusive, but the operator $T \in \mathcal{L}(L_1)$ defined by $Tx = x \cdot \mathbf{1}_{[0,1/2]}$ for each $x \in L_1$, is an example of an operator which is neither a sign-embedding nor narrow.

A natural question concerning Definition 8.1 is whether one could equivalently relax the requirement that (8.1) holds only for mean zero signs instead of all signs, similarly as it can be done in the definition of a narrow operator (Proposition 1.9).

We show that the answer is negative (Example 8.11). However, if an operator $T \in \mathcal{L}(L_1, X)$ (not necessarily injective) satisfies (8.1) for each mean zero sign then there exists a subspace E_1 of L_1, isometric to L_1, so that the restriction $T|_{E_1}$ is a sign-embedding (see Proposition 8.6).

Semi-embeddings

The following weak version of the notion of an embedding was introduced by Lotz, Peck and Porta [84] (1979), and was investigated by several mathematicians, e.g. Bourgain, Drewnowski, Fonf, Ghoussoub, Maurey, Rosenthal, etc.

Definition 8.2. Let X and Y be Banach spaces. An injective operator $T \in \mathcal{L}(X, Y)$ is called a *semi-embedding* if TB_X is closed.

One of the main results of Bourgain and Rosenthal [20] (1983) states that, if a separable Banach space semi-embeds in a Banach space with the Radon–Nikodým property (RNP, in short) then it itself has the RNP. In particular, L_1 does not semi-embed in a Banach space with the RNP. In fact, for L_1 we have the following stronger result which follows directly from the definitions: L_1 does not semi-embed in a Banach space with the *Krein–Milman property* (KMP, in short) (recall that a Banach space X has the KMP if every nonempty closed bounded convex subset of X is the closed convex hull of its extremal points). This result is stronger because the RNP implies the KMP by a classical theorem of Lindenstrauss.

Bourgain and Rosenthal in [20] showed that the restriction of a semi-embedding to a subspace of the domain space need not be a semi-embedding. This motivated the definition of a weaker type of embedding, which is however inherited by restrictions to a subspace.

G_δ-embeddings

The following notion was introduced by Bourgain and Rosenthal in [20] (1983).

Definition 8.3. An injective operator $T \in \mathcal{L}(X, Y)$ is called a G_δ-*embedding* if TK is a G_δ-set for each closed bounded $K \subset X$.

Ghoussoub and Rosenthal studied G_δ-embeddings in [44] and proved in particular that a G_δ-embedding $T \in \mathcal{L}(L_1, X)$ cannot be narrow, which is the strongest result in the direction that a G_δ-embedding cannot be a "small" operator, see Theorem 8.16 and Section 8.4.

Properties

Fonf showed in [42] that for a Banach space X the following assertions are equivalent:

(i) X contains no subspace isomorphic to a conjugate space.

(ii) Each semi-embedding from X to a Banach space Y is an into isomorphism.

(iii) Each G_δ-embedding from X to a Banach space Y is an into isomorphism.

Bourgain and Rosenthal [20] showed that every semi-embedding is a G_δ-embedding. Apart from this, there are no relationships between the three notions of embeddings. Below we show examples of the following embeddings of L_1:

- a G_δ-embedding which is not a semi-embedding (Example 8.12);
- a semi-embedding which is not a sign-embedding (Example 8.13);
- a sign-embedding which is not a G_δ-embedding (Example 8.15).

8.2 Embeddability of L_1

We say that L_1 *semi-embeds* (resp., *sign-embeds*, or G_δ*-embeds*) in a Banach space X provided there exists a semi-embedding (resp., sign-embedding, or G_δ-embedding) $T \in \mathcal{L}(L_1, X)$.

Rosenthal [127] proved the following characterization of non-sign-embeddability of L_1.

Theorem 8.4. *For any Banach space X the following two assertions are equivalent:*

(i) *L_1 does not sign-embed in X.*

(ii) *Every operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Proof. Note that (ii) trivially implies (i).

To prove the converse implication, suppose that $T \in \mathcal{L}(L_1, X)$ is not narrow. By Theorem 7.64, T is not somewhat narrow, that is, there exists $A \in \Sigma^+$ so that T satisfies (8.1) for every sign on A. This is almost a sign-embedding that we are looking for, however T does not have to be injective. By a standard argument, which we show below as a lemma, for easy reference, there exists an atomless sub-σ-algebra $\mathcal{F}$ of Σ, so that T restricted to $L_1(\mathcal{F})$ is one-to-one. Since there exists an isometry $V : L_1 \to L_1(\mathcal{F})$ so that Vx is a sign, whenever x is a sign, the operator TV is the desired sign-embedding from L_1 into X. □

Lemma 8.5. *Let X be any Banach space and $T \in \mathcal{L}(L_1, X)$. Then there exists an atomless sub-σ-algebra $\mathcal{F}$ of Σ, so that T restricted to $L_1(\mathcal{F})$ is one-to-one.*

Proof. By induction on n, we will construct measurable sets $E_n \subseteq [0,1]$, and finite subsets F_n of the unit ball of X^* so that for all $n \in \mathbb{N}$

(i) $E_n = E_{2n} \cup E_{2n+1}$ and $E_{2n} \cap E_{2n+1} = \emptyset$;

(ii) $\mu(E_{2n}) = \mu(E_{2n+1}) = \frac{1}{2}\mu(E_n)$;

(iii) if $h_0 = \mathbf{1}_{[0,1]}$, $h_n = \mathbf{1}_{E_{2n}} - \mathbf{1}_{E_{2n+1}}$ and $Y_n = \mathrm{span}\{h_j\}_{j=0}^n$, then for every $x \in TY_n$

$$\max\{|f(x)| : f \in F_n\} \geq \frac{1}{2}\|x\|,$$

(iv) $f(Th_n) = 0$ for all $f \in F_{n-1}$;

(v) $F_{n-1} \subset F_n$.

To start the construction, let $E_1 = [0, 1]$, $h_0 = \mathbf{1}_{[0,1]}$, f be an element of the unit ball of X^* so that $f(Th_0) = \|Th_0\|$ and $F_0 = \{f\}$.

If $\{h_j\}_{j=0}^{n-1}$, E_n and F_{n-1} have been constructed then, by the Lyapunov convexity theorem, there exists a measurable set $E = E_{2n}$ so that $E \subset E_n$, $\mu(E) = \frac{1}{2}\mu(E_n)$ and $\int_E f\, d\mu = \frac{1}{2}\int_{E_n} f\, d\mu$ for all $f \in T^* F_{n-1}$. Let $E_{2n+1} = E_n \setminus E_{2n}$. This defines h_n, and by the choice of E_n, $f(Th_n) = 0$ for all $f \in F_{n-1}$. We choose $f_n \notin F_{n-1}$ so that (iii) holds, and put $F_n = F_{n-1} \cup \{f_n\}$, so the inductive step of the construction is completed.

Let $\mathcal{F}$ be the sub-σ-algebra generated by the sets $\{E_n\}_{n=1}^{\infty}$. It follows that $\{h_n\}_{n=0}^{\infty}$ is a basis for $L_1(\mathcal{F})$ and $(Th_n)_{n=0}^{\infty}$ is a basic sequence in X, so $T\big|_{L_1(\mathcal{F})}$ is one-to-one. □

Using a very similar idea we also obtain.

Proposition 8.6. *Suppose $T \in \mathcal{L}(L_1, X)$ satisfies (8.1) for some $\delta > 0$ and each mean zero sign x. Then there exists a subspace E_1 of L_1, isometric to L_1, so that the restriction $T|_{E_1}$ is a sign-embedding.*

Proof. Consider the map $S : L_1[0, 1] \to L_1[0, 1]$ given by

$$Sx(t) = \begin{cases} -x(1-2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ x(2t-1) & \text{if } \frac{1}{2} < t < 1. \end{cases}$$

Then for all $x \in L_1[0, 1]$, $\|Sx\|_1 = \|x\|_1$, $\int_{[0,1]} Sx\, d\mu = 0$, and Sx is a sign if x is a sign. Thus, for all signs x, we have $\|TSx\|_1 \ge \delta\|x\|_1$. By Lemma 8.5, there exists an atomless sub-σ-algebra $\mathcal{F}$ of Σ, so that TS restricted to $L_1(\mathcal{F})$ is one-to-one. Since S is an isometry, $E_1 = S^{-1}(L_1(\mathcal{F}))$ is isometric to L_1, and $T\big|_{E_1}$ is a sign-embedding. □

Rosenthal [126] showed that, if L_1 G_δ-embeds in X then L_1 sign-embeds in X (see Corollary 8.18 below). We do not know whether there any relations for other types of embeddabilities.

Open problem 8.7.

(a) Suppose that L_1 sign-embeds in X. Does L_1 G_δ-embed in X?

(b) Suppose that L_1 G_δ-embeds in X. Does L_1 sign-embed in X?

(c) Suppose that L_1 sign-embeds in X. Does L_1 semi-embed in X?

Each of the notions of weak embeddings naturally leads to the question whether it is in fact weaker than the isomorphic embedding. That is, assuming that L_1 semi-(G_δ, or sign)-embeds in X, does it imply that L_1 embeds isomorphically in X?

For semi-embeddings and G_δ-embeddings this problem was posed by Bourgain and Rosenthal in [20] and for sign-embeddings by Rosenthal in [127]. Since, as mentioned above, G_δ- and semi-embeddings of L_1 are weaker than sign-embeddings, a negative answer to the problem concerning sign-embeddings will imply the same answer to the problem concerning G_δ-embeddings and semi-embeddings. Thus, we will concentrate on the following problem posed by Rosenthal.

Problem 8.8. Suppose L_1 sign-embeds in X. Does L_1 embed isomorphically in X?

It follows from Talagrand's work on the three-space problem for L_1 [138], that the answer to Rosenthal's problem is negative, see Corollary 8.25 below. Thus sign-embeddability, semi-embeddability and G_δ-embeddability of L_1 are all distinct from isomorphic embeddability of L_1. However, Ghoussoub and Rosenthal [44] showed that they do coincide for embeddings into a large class of Banach spaces, including separable dual spaces, see Corollary 8.23 below.

Rosenthal [126] showed that sign-embeddings satisfy the following three-space property.

Theorem 8.9. *If L_1 sign-embeds in X and Y is a subspace of X then L_1 sign-embeds either in Y or in X/Y.*

Proof. Let $T : L_1 \to X$ be a sign-embedding, and let $\tau : X \to X/Y$ be the quotient map. Choose $\delta > 0$ so that $\|Tx\| \ge \delta\|x\|$ for every sign $x \in L_1$. Assume that $S = \tau T : L_1 \to X/Y$ is not a sign-embedding. In other words, S is somewhat narrow. By Theorem 7.64, T is narrow. By Theorem 2.21, there exists an L_1-normalized Haar-type system (g_n) on $[0,1]$ so that $\|Sg_n\| < 2^{-n-2}\delta$ for each $n \in \mathbb{N}$. Hence, by the definition of a quotient map, for each $n \in \mathbb{N}$ there exists $y_n \in Y$ so that $\|Tg_n - y_n\| < 2^{-n-2}\delta$. Let Σ_1 be the sub-σ-algebra of Σ generated by (g_n). Define an operator $T_1 \in \mathcal{L}(L_1(\Sigma_1), Y)$ by setting $T_1 g_n = y_n$ for all $n \in \mathbb{N}$ and extending by linearity and continuity to the entire domain space. It is a standard exercise to show that T_1 is well defined and bounded. Observe that for every $x = \sum_{n=1}^\infty a_n g_n \in L_1(\Sigma_1)$ we have

$$\|(T - T_1)x\| = \Big\|\sum_{n=1}^\infty a_n(Tg_n - y_n)\Big\| \le 2\|x\| \sum_{n=1}^\infty \|Tg_n - y_n\| < \frac{\delta}{2}\|x\|$$

(we use here that $|a_n| \le 2\|x\|$ for each $n \in \mathbb{N}$ [79, p. 7]). Therefore, $\|T_1 - T\| \le \delta/2$. Hence, for every sign $x \in L_1(\Sigma_1)$

$$\|T_1 x\| \ge \|Tx\| - \|T_1 - T\|\|x\| \ge \delta\|x\| - \frac{\delta}{2}\|x\| = \frac{\delta}{2}\|x\|\,.$$

Thus, T_1 is a sign-embedding. Finally, if J is the isometry of L_1 onto $L_1(\Sigma)$ that sends the usual L_1-normalized Haar system on $[0,1]$ to (g_n), then $T_1 J$ is a sign-embedding of L_1 to Y. □

Talagrand [138] proved that the isomorphic embeddings do not have the three-space property (see Theorem 8.24). We do not know whether the notions of semi- or G_δ-embeddings have the three-space property like sign-embeddings.

Open problem 8.10.

(a) Assume that L_1 semi-embeds in X, and Y is a subspace of X. Does L_1 semi-embed either in Y or in X/Y?

(b) Assume that L_1 G_δ-embeds in X, and Y is a subspace of X. Does L_1 G_δ-embed either in Y or in X/Y?

There is one common property of semi, G_δ and sign-embeddings: Theorems 7.2 and 8.4 imply that every sign-embedding and every G_δ-embedding (hence, each semi-embedding) $T : L_1 \to X$ to any Banach space X fixes a copy of ℓ_1 [20].

8.3 Examples

First, we answer negatively the question whether we could require that (8.1) holds only for mean zero signs in the definition of a sign-embedding of L_1.

Example 8.11. There exists an injective operator $T \in \mathcal{L}(L_1)$ such that (8.1) holds for some $\delta > 0$ and every mean zero sign $x \in L_1$ but which is not a sign-embedding.

Proof. Let $[0,1] = \bigsqcup_{n=1}^{\infty} A_n$ with $\mu(A_n) > 0$. Let $(\alpha_n)_{n=1}^{\infty}$ be a bounded sequence of scalars. For each $n \geq 1$, define $T_n \in \mathcal{L}(L_1(A_n))$ by setting for each $x \in L_1$

$$T_n x = \frac{\alpha_n}{\mu(A_n)} \left(\int_{A_n} x \, \mathrm{d}\mu \right) \mathbf{1}_{A_n} + \left(x - \frac{1}{\mu(A_n)} \left(\int_{A_n} x \, \mathrm{d}\mu \right) \mathbf{1}_{A_n} \right).$$

Note that $\|T_n\| \leq |\alpha_n| + 2$. Define an operator $T \in \mathcal{L}(L_1)$ by putting for each $x \in L_1$

$$Tx = \sum_{n=1}^{\infty} T_n \big(x \cdot \mathbf{1}_{A_n} \big).$$

Thus, we have that $\|T\| \leq \sup_n |\alpha_n| + 2$. If $\alpha_n \neq 0$ for some n, then T_n and hence T is injective.

We claim that if $\inf_n |\alpha_n| = 0$ and $\sup_n |\alpha_n| < \frac{1}{5}$ then T satisfies the conditions of Example 8.11.

Indeed, for each n we have $T\mathbf{1}_{A_n} = T_n \mathbf{1}_{A_n} = \alpha_n \mathbf{1}_{A_n}$. Thus, if $\inf_n |\alpha_n| = 0$ then T is not a sign-embedding.

Next, suppose that $\mu(A_n) = \frac{3}{4^n}$ and $\sup_n |\alpha_n| = \alpha < \frac{1}{5}$. We claim that then (8.1) holds for some $\delta > 0$ and every mean zero sign $x \in L_1$.

To see this, fix any mean zero sign $x \in L_1$ and set $B = \operatorname{supp} x$, $B_n = B \cap A_n$, $x_n = x \cdot \mathbf{1}_{B_n}$ and

$$\gamma_n = \frac{1}{\mu(A_n)} \int_{A_n} x \, \mathrm{d}\mu$$

for each n. Observe that

$$\|x\| = \mu(B) = \sum_{n=1}^{\infty} \mu(B_n) \text{ and } \|x_n\| = \mu(B_n).$$

Put $M = \{n : |\gamma_n| \le \frac{5}{6}\}$ and show that

$$\sum_{n \in M} \mu(B_n) \ge \frac{1}{5} \mu(B) . \tag{8.2}$$

Supposing the contrary, we obtain

$$\sum_{n \notin M} \mu(B_n) > \frac{4}{5} \mu(B). \tag{8.3}$$

If we put $n_0 = \min(\mathbb{N} \setminus M)$ then

$$\begin{aligned}\sum_{n \notin M} \mu(B_n) \le \sum_{n \ge n_0} \mu(B_n) \le \sum_{n \ge n_0} \mu(A_n) \\ = \mu(A_{n_0})\Big(1 + \frac{1}{4} + \frac{1}{16} + \dots\Big) = \frac{4}{3} \mu(A_{n_0}).\end{aligned} \tag{8.4}$$

Combining (8.4) and (8.3), we get

$$\mu(A_{n_0}) \ge \frac{3}{4} \sum_{n \notin M} \mu(B_n) > \frac{3}{5} \mu(B). \tag{8.5}$$

Since $|\gamma_{n_0}| > \frac{5}{6}$, we have that

$$\Big|\int_{A_{n_0}} x \, \mathrm{d}\mu\Big| > \frac{5}{6} \mu(A_{n_0}) > \frac{1}{2} \mu(B) . \tag{8.6}$$

Since x is a mean zero sign, we have that

$$\Big|\int_A x \, \mathrm{d}\mu\Big| \le \frac{1}{2} \mu(B)$$

for each measurable $A \subseteq B$, that contradicts (8.6). Thus, (8.2) is proved.

Suppose $t \in B_n$. Then

$$(Tx)(t) = (Tx_n)(t) = (T_n x_n)(t) = \alpha_n \gamma_n + x_n(t) - \gamma_n = x_n(t) - \gamma_n(1-\alpha_n)\,.$$

If $n \in M$ and $t \in B_n$ then

$$\begin{aligned} |(Tx)(t)| &\geq |x_n(t)| - |\gamma_n| \cdot |1-\alpha_n| \geq |x_n(t)| - |\gamma_n| \cdot (1+|\alpha_n|) \\ &\geq 1 - \frac{5}{6}(1+\alpha) = \frac{1}{6} - \frac{5\alpha}{6} = \frac{1-5\alpha}{6}. \end{aligned}$$

Hence

$$\int_{B_n} |Tx| \,\mathrm{d}\mu \geq \frac{1-5\alpha}{6}\mu(B_n)\,,$$

and by (8.2)

$$\begin{aligned} \|Tx\| &\geq \sum_{n\in M} \int_{B_n} |Tx| \,\mathrm{d}\mu \geq \frac{1-5\alpha}{6} \sum_{n \in M} \mu(B_n) \\ &\geq \frac{1-5\alpha}{30}\,\mu(B) = \frac{1-5\alpha}{30}\|x\|. \end{aligned}$$

Thus, (8.1) holds for $\delta = \frac{1-5\alpha}{30}$ and every mean zero sign $x \in L_1$. □

Example 8.12. There is a G_δ-embedding $T \in \mathcal{L}(L_1)$ which is not a semi-embedding.

Proof. Let $(A_n)_{n=1}^\infty$ be any sequence of disjoint elements of Σ^+. For each $n \geq 1$, we set $e_n = \frac{\mathbf{1}_{A_n}}{\mu(A_n)}$. Let X be the closed linear span of $\{e_n\}_n$ in L_1, and Y be the natural complement to X in L_1, i.e.

$$Y = \left\{x \in L_1 : (\forall n \geq 1) \int_{A_n} x \,\mathrm{d}\mu = 0\right\}.$$

Let $(d_n)_{n\geq 1}$ be a sequence of positive reals with $d_n \searrow 0$. For $(a_k)_{k\in\mathbb{N}}$ in ℓ_1 and $y \in Y$ we define the following:

$$T\Big(\sum_{k=1}^\infty a_k e_k + y\Big) = \Big(\sum_{k=1}^\infty a_k\Big) e_1 + \sum_{k=2}^\infty a_{k-1} d_{k-1} e_k + y\,.$$

Since $(e_n)_{n=1}^\infty$ is isometrically equivalent to the unit vector basis of ℓ_1 and by the definition of Y, T is a well-defined operator on L_1.

We show that T has the desired properties.

Since $e_1 \notin TL_1$ and $Te_n = e_1 + d_n e_{n+1} \to e_1$ as $n \to \infty$, we get that T is not a semi-embedding.

To prove that T is a G_δ-embedding, consider the Banach space $E = \ell_1 \oplus_1 L_1$. Define the operator $S \in \mathcal{L}(E)$ by $S(\widetilde{e}_n, y) = (d_n \widetilde{e}_n, y)$, where $(\widetilde{e}_n)_{n=1}^\infty$ is the unit vector basis of ℓ_1 and $y \in L_1$, and extend to E by linearity and continuity. It is routine to check that S is a semi-embedding. In particular, S is a G_δ-embedding.

Define $U \in \mathcal{L}(L_1)$ by

$$U\Big(\sum_{k=1}^{\infty} a_k e_k + y\Big) = \sum_{k=1}^{\infty} a_{k+1} e_k + y\,,$$

where $y \in Y$. Denote $S_1 = U \circ T$. Then

$$S_1\Big(\sum_{k=1}^{\infty} a_k e_k + y\Big) = \sum_{k=1}^{\infty} a_k d_k e_k + y\,,$$

where $y \in Y$. It is an easy observation that E is isomorphic to L_1 and operators S and S_1 are isomorphically equivalent (i.e. $S_1 = J^{-1} \circ S \circ J$ for some isomorphism $J : L_1 \to E$). Hence, S_1 is a G_δ-embedding.

Let K be any closed bounded subset of L_1. We will show that $M = TK$ is a G_δ set in L_1, and thus that T is a G_δ-embedding.

Since S_1 is a G_δ-embedding, we can write $M_1 = S_1 K = \bigcap_{n=1}^{\infty} G_n$, where G_n are open sets in L_1 for all $n \in \mathbb{N}$. Fix any $n \in \mathbb{N}$ and any

$$z = \sum_{k=1}^{\infty} \eta_k e_k + y \in M\,.$$

Since $z \in TL_1$, we have $\eta_1 = \sum_{k=1}^{\infty} d_k^{-1} \eta_{k+1}$. Let $k > n$ so that

$$\Big|\eta_1 - \sum_{i=1}^{k} d_i^{-1} \eta_{i+1}\Big| < \frac{1}{n}\,.$$

By continuity of the coordinate functionals in ℓ_1, there exists a neighborhood $V_z^{(n)}$ of z open in L_1 such that for each $v = \sum_{k=1}^{\infty} \xi_k e_k + y \in V_z^{(n)}$,

$$\Big|\xi_1 - \sum_{i=1}^{k} d_i^{-1} \xi_{i+1}\Big| < \frac{1}{n}\,.$$

Let

$$V_n = \Big(\bigcup_{z \in M} V_z^{(n)}\Big) \cap U^{-1}(G_n)\,.$$

Since $M_1 = UM \subseteq G_n$, we obtain that $M \subseteq U^{-1}(G_n)$ and hence, $M \subseteq V_n$.

We will show that $M = \bigcap_{n=1}^{\infty} V_n$. Indeed, the inclusion $M \subseteq \bigcap_{n=1}^{\infty} V_n$ is already shown. To prove the converse, let

$$z_0 = \sum_{k=1}^{\infty} \eta_k e_k + y \in \bigcap_{n=1}^{\infty} V_n\,.$$

Then $Uz_0 \in G_n$, for all $n \in \mathbb{N}$. Therefore, $z_0 \in \bigcap_{n=1}^{\infty} G_n = M_1$. Thus,

$$x_0 = \sum_{k=1}^{\infty} d_k^{-1} e_{k+1} + y \in K \,.$$

Fix any $n \in \mathbb{N}$. Since $z_0 \in V_n$, there is $z \in M$ such that $z_0 \in V_z^{(n)}$. Then there exists $k_n > n$ so that

$$\left| \eta_1 - \sum_{i=1}^{k_n} d_i^{-1} \eta_{i+1} \right| < \frac{1}{n} \,.$$

Passing to the limit as $n \to \infty$, we obtain $\eta_1 = \sum_{i=1}^{\infty} d_i^{-1} \eta_{i+1}$, i.e. $z_0 = T x_0$. Hence, M is a G_δ set in L_1. Thus, T is a G_δ-embedding. □

Example 8.13. There is a semi-embedding $T \in \mathcal{L}(L_1)$ that is not a sign-embedding.

Proof. Decompose $[0,1] = \bigsqcup_{k=1}^{\infty} A_k$, where $A_k \in \Sigma^+$, and for each $x \in L_1$ put

$$Tx = \sum_{k=1}^{\infty} \frac{1}{k} \left(\mathbf{1}_{A_k} \cdot x \right) .$$

T is not a sign-embedding, since $\|T \mathbf{1}_{A_k}\| = k^{-1} \, \|\mathbf{1}_{A_k}\|$.

It is an easy technical exercise to prove that T is a semi-embedding. Indeed, the injectivity of T is obvious. Let $x_n \in B_{L_1}$ be elements with $\lim_{n\to\infty} T x_n = y$, i.e.

$$y = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k} \left(\mathbf{1}_{A_k} \cdot x_n \right) .$$

For each $k \in \mathbb{N}$, let P_k be the projection $P_k x = \mathbf{1}_{A_k} \cdot x$. Then

$$P_k y = \lim_{n\to\infty} \frac{1}{k} P_k x_n , \quad \text{i.e.} \quad \lim_{n\to\infty} P_k x_n = k \, P_k y \,.$$

Let $x = \sum_{k=1}^{\infty} k \, (\mathbf{1}_{A_k} \cdot y)$. We claim that $\sum_{k=1}^{\infty} k \, \|\mathbf{1}_{A_k} \cdot y\| \le 1$. Indeed, if not, then there exists m so that $\sum_{k=1}^{m} k \, \|\mathbf{1}_{A_k} \cdot y\| > 1$, and since $\lim_{n\to\infty} T x_n = y$, there exists n with $\sum_{k=1}^{m} k \, \|\mathbf{1}_{A_k} \cdot T x_n\| > 1$. But

$$\mathbf{1}_{A_k} \cdot T x_n = \frac{1}{k} \mathbf{1}_{A_k} \cdot x_n$$

and

$$\sum_{k=1}^{\infty} k \, \|\mathbf{1}_{A_k} \cdot T x_n\| = \sum_{k=1}^{\infty} \|\mathbf{1}_{A_k} \cdot x_n\| = \|x_n\| \le 1 \,,$$

which is a contradiction. □

Example 8.14. There exists a projection $Q \in \mathcal{L}(L_1)$ with the following properties:

(i) $\|Qx\| \geq \delta \|x\|$ for some $\delta > 0$ and each sign $x \in L_1$.

(ii) Both the range and the kernel of Q are isomorphic to L_1.

Proof. We define an operator $J : L_1[0,1/2] \to L_1[1/2,1]$ by setting for each $x \in L_1[0,1/2]$

$$(Jx)(t) = 2x(1-t), \quad t \in [0,1] .$$

Then for each $x \in L_1$ we set

$$Qx = x \cdot \mathbf{1}_{[1/2,1]} - J\big(x \cdot \mathbf{1}_{[0,1/2)}\big) .$$

Evidently, Q is a projection of L_1 onto $L_1[1/2,1]$. Since $L_1[0,1/2]$ is a complement to $L_1[1/2,1]$ which is isomorphic to L_1, we obtain that $\ker Q$, being another complement to $L_1[1/2,1]$, is also isomorphic to L_1.

To prove (i), we consider any sign $x \in L_1$, say $x^2 = \mathbf{1}_C$ for some $C \in \Sigma$. Then we set

$$y = x \cdot \mathbf{1}_{[0,1/2)}, \quad z = x \cdot \mathbf{1}_{[1/2,1]} ,$$
$$A = \big\{t \in [1/2,1] : (Jy)(t) = 0\big\}, \quad B = \big\{t \in [1/2,1] : z(t) = 0\big\} .$$

Note that z takes values ± 1 on $[1/2,1] \setminus B$ and 0 on B and Jy takes values ± 2 on $[1/2,1] \setminus A$ and 0 on A. Hence,

$$\begin{aligned}\|Qx\| = \|z - Jy\| &= \int_{[1/2,1]\setminus(A\cap B)} |z - Jy| \,\mathrm{d}\mu \geq \mu\big([1/2,1] \setminus (A \cap B)\big) \\ &= \frac{1}{2} - \mu(A \cap B) \geq \max\Big\{\frac{1}{2} - \mu(A),\ \frac{1}{2} - \mu(B)\Big\}.\end{aligned}$$

Denote $C_1 = C \cap [0,1/2)$ and $C_2 = C \cap [1/2,1]$. Since $\mu(C_1) = \frac{1}{2} - \mu(A)$ and $\mu(C_2) = \frac{1}{2} - \mu(B)$, we obtain

$$\|Qx\| \geq \max\big\{\mu(C_1),\ \mu(C_2)\big\} \geq \frac{\mu(C)}{2} = \frac{\|x\|}{2} .$$

□

Example 8.15. There exists a sign-embedding $T \in \mathcal{L}(L_1)$ which is not a G_δ-embedding (and hence not a semi-embedding).

Proof. Let Q be a projection of L_1 from the previous example and $L_1 = X \oplus Y$ be the corresponding decomposition with $X = Q(L_1)$ and $Y = \ker Q$. Let $S \in \mathcal{L}(Y)$ be any injective compact operator and set

$$T = Q + S \cdot (I - Q) .$$

We claim that T satisfies the desired properties. Indeed, if $x \in L_1$ is a sign then

$$\begin{aligned}\|Tx\| &= \|Qx + S \cdot (I - Q)x\| \geq \|Q\|^{-1} \big\|Q \cdot \big(Qx + S \cdot (I - Q)x\big)\big\| \\ &= \|Q\|^{-1} \|Qx\| \geq \|Q\|^{-1} \delta \|x\|.\end{aligned}$$

Now we show that T is injective. Suppose $Tx = 0$. Then $Qx = 0$ and $S \cdot (I - Q)x = 0$, because

$$Q(L_1) \cap S \cdot (I - Q)(L_1) \subseteq X \cap Y = \{0\}\,.$$

Since $Qx = 0$, we have that $x \in Y$ and $0 = S \cdot (I - Q)x = Sx$. By injectivity of S, we obtain $x = 0$.

It remains to show that T is not a G_δ-embedding. Since S is compact and hence narrow, it could not be a G_δ-embedding (see Corollary 8.16 below). Thus there exists a closed bounded subset $K \subseteq Y$ so that SK is not a G_δ set. But $TK = SK$ and thus T is not a G_δ-embedding. □

8.4 G_δ-embeddings of L_1 are not narrow

This section is devoted to the proof of the following theorem.

Theorem 8.16. *Let Y be a Banach space. Then any G_δ-embedding $S \in \mathcal{L}(L_1, Y)$ is not narrow.*

This is a consequence of the following more general result of Ghoussoub and Rosenthal [44].

Theorem 8.17. *Let S be a G_δ-embedding of a Banach space X into a Banach space Y. Then for any $T \in \mathcal{L}(L_1, X)$ the operator T is narrow if and only if $ST \in \mathcal{L}(L_1, Y)$ is narrow.*

Indeed, Theorem 8.16 is Theorem 8.17 applied to $X = L_1$ and T the identity operator on L_1.

Theorem 8.16 combined with Theorem 8.4, immediately gives the following result.

Corollary 8.18. *If L_1 G_δ-embeds in a Banach space X then L_1 sign-embeds in X.*

To prove Theorem 8.17, we need several lemmas and definitions.

Trees and martingales

Let X be a Banach space. A sequence $(x_{n,k})_{n=1,k=1}^{\infty \ \ 2^n}$ in X is called a *tree* if $2x_{n,k} = x_{n+1,2k-1} + x_{n+1,2k}$, for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$. Note that if $(E_{n,k})_{n=1,k=1}^{\infty \ \ 2^n}$

is a tree of sets in the sense of Definition 1.3 then $x_{n,k} = \frac{\mathbf{1}_{E_{n,k}}}{\mu(\mathbf{1}_{E_{n,k}})}$ is a tree in any Köthe–Banach space on [0, 1].

Let $(\mathcal{F}_\alpha)$ be an increasing net of sub-σ-algebras of Σ. A net (f_α) in $L_1(X)$ with the same index set is called a *martingale* with respect to $(\mathcal{F}_\alpha)$ if for every $\alpha \leq \beta$, $M^{\mathcal{F}} f_\beta = f_\alpha$. Taking $\alpha = \beta$ in the definition, we obtain that f_α is $\mathcal{F}_\alpha$-measurable for each α. Since $M^{\mathcal{F}}$ is a contractive projection, $\|f_\alpha\|_{L_p(X)} \leq \|f_\beta\|_{L_p(X)}$ for every $\alpha \leq \beta$ and $p \geq 1$.

Let $\mathcal{F}_n$ for $n = 0, 1, \ldots$, be the algebra generated by the dyadic intervals $(I_n^k)_{k=1}^{2^n}$. An X-valued martingale $(f_n)_{n=0}^\infty$ with respect to $(\mathcal{F}_n)_{n=0}^\infty$ is called a *dyadic martingale*. There is a close connection between dyadic martingales and trees. Indeed, for each tree $(x_{n,k})_{n=0,k=1}^{\infty\ \ 2^n}$ in X the sequence $f_n = \sum_{k=1}^{2^n} x_{n,k} \mathbf{1}_{I_n^k}$, $n = 0, 1, \ldots$ is a dyadic martingale. Conversely, if $(f_n)_{n=0}^\infty$ is an X-valued dyadic martingale then the sequence $x_{n,k} = 2^n \int_{I_n^k} f_n \, \mathrm{d}\mu$, $n = 0, 1, \ldots, k = 1, \ldots, 2^n$, is a tree in X.

Lemma 8.19. *Let X, Y be Banach spaces, $S \in \mathcal{L}(X, Y)$, (Ω, Σ, μ) a finite measure space, and $K \subset X$ a closed bounded separable convex set. Suppose that $S(K)$ is a G_δ-set and $S|_K$ is an injective map. If (f_n) is a K-valued martingale with respect to a sequence (Σ_n) of sub-σ-algebras of Σ such that (Sf_n) converges a.e. to some $S(K)$-valued function $g \in L_1(Y)$, then (f_n) converges a.e. to $S^{-1}g$.*

Proof. Let $f = S^{-1}g$. Since K is a Polish space and $S|_K$ is injective and continuous, it follows from the selection theorem [26, Theorem 8.5.3] that f is measurable. Since K is bounded, so is f, and hence, $f \in L_1(X)$. Fix any $n \in \mathbb{N}$. Passing to a limit as $m \to \infty$ in the equality $M^{\Sigma_n} f_m = f_n$ which holds for every $m \geq n$, we obtain $M^{\Sigma_n} f = f_n$. By the Doob martingale convergence theorem, $f_n \to f$ a.e. on Ω. □

An L_1-valued dyadic martingale $(f_n)_{n=0}^\infty$ is called a *standard dyadic martingale* if there exists a tree of sets $(E_{n,k})_{n=1,k=1}^{\infty\ \ 2^n}$ in the sense of Definition 1.3 such that

$$f_n(t) = 2^n \sum_{k=1}^{2^n} \mathbf{1}_{E_{n,k}} \mathbf{1}_{I_n^k}(t), \quad t \in [0,1], \tag{8.7}$$

that is, the corresponding tree is $e_{n,k} = \frac{\mathbf{1}_{E_{n,k}}}{\|\mathbf{1}_{E_{n,k}}\|}$.

Let $\mathcal{P} = \{x \in S_{L_1} : x \geq 0\}$ be the positive face of L_1.

Lemma 8.20. *Let X be a Banach space and $T \in \mathcal{L}(L_1, X)$ be a narrow operator. Let G be a G_δ-set in X such that $T\mathcal{P} \subseteq G$. Then there exists a standard dyadic martingale $(f_n)_{n=0}^\infty$ such that $(Tf_n)_{n=0}^\infty$ converges a.e. to a function $g \in L_1(X)$ with values in G.*

Proof. Let $U_1 \supseteq U_2 \supseteq \dots$ be open sets in X so that $G = \bigcap_{n=1}^{\infty} U_n$. We construct recursively a tree of sets $(E_{n,k})_{n=1,k=1}^{\infty\ \ 2^n}$ with $E_{0,1} = [0,1]$, and a sequence of open balls $(B_{n,k})_{n=1,k=1}^{\infty\ \ 2^n}$ in X such that for every $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$

(a) $B_{n,k}$ is centered at the point $T(\frac{\mathbf{1}_{E_{n,k}}}{\|\mathbf{1}_{E_{n,k}}\|}) = 2^n T\mathbf{1}_{E_{n,k}}$ and has radius $0 < r_n \le 1/n$ and $0 < r_n < r_{n-1} < \dots$.

(b) $B_{n,k} \subseteq U_{2^n+k}$ and $\overline{B}_{n+1,2k-1} \cup \overline{B}_{n+1,2k} \subseteq \overline{B}_{n,k}$.

At the first step we choose an open ball $B_{0,1} \subseteq U_1$ centered at $T\mathbf{1}_{[0,1]}$ with radius $0 < r_0 \le 1/2$. Since T is narrow, we can partition $E_{0,1} = E_{1,1} \sqcup E_{1,2}$ so that $\mu(E_{0,1}) = \mu(E_{1,1}) = 1/2$ and $\|Tx\| < r_0/2$, where $x = \mathbf{1}_{E_{1,1}} - \mathbf{1}_{E_{1,2}}$. Then for $j = 1, 2$

$$\left\|2T\mathbf{1}_{1,j} - T\mathbf{1}_{[0,1]}\right\| = \left\|2T\mathbf{1}_{1,j} - T(\mathbf{1}_{1,1} + \mathbf{1}_{1,2})\right\| = \|Tx\| < r_0/2\,.$$

Hence we can find $0 < r_1 < r_0$ and $r_1 \le 1/8$, so that the balls $B_{1,1}$ and $B_{1,2}$ defined by (a) satisfy (b). We continue the construction inductively to obtain the desired sequences.

Let $(f_n)_{n=0}^{\infty}$ be the standard dyadic martingale with respect to the constructed above tree $(E_{n,k})_{n=1,k=1}^{\infty\ \ 2^n}$ defined by (8.7). Given any $t \in [0,1]$ and $n \in \mathbb{N}$, let $k_n \in \{1, \dots, 2^n\}$ be such that $t \in E_{n,k_n}$. Since r_n tends to zero, by (b) there exists a unique point $g(t) \in \bigcap_{n=1}^{\infty} \overline{B}_{n,k_n} \subset G$. Since $f_n(t) = 2^n \mathbf{1}_{E_{n,k_n}}$, we have that $(Tf_n)(t) \in B_{n,k_n}$ for each $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty}(Tf_n)(t) = g(t)$. Since it is a limit of a sequence of simple functions, $g \in L_1(X)$. □

Lemma 8.21. *Let (f_n) be a standard dyadic martingale, X a Banach space, and $T \in \mathcal{L}(L_1, X)$ a generalized sign-embedding. Then at no point does (Tf_n) converge.*

Proof. Let $\delta > 0$ be such that $\|Tx\| \ge \delta\|x\|$ for each sign x. Let $(E_{n,k})$ be the tree of sets such that (8.7) holds. Fix any $t \in [0,1]$. For each $n \in \mathbb{N}$, let $k_n \in \{1, \dots, 2^n\}$ be the number such that $t \in E_{n,k_n}$. Then $(Tf_n)(t) = 2^n T\mathbf{1}_{E_{n,k_n}}$ and $(Tf_{n+1})(t) = 2^{n+1}T\mathbf{1}_{E_{n+1,k_{n+1}}}$. Choose $\theta \in \{-1, 1\}$ so that

$$\mathbf{1}_{E_{n,k_n}} = \mathbf{1}_{E_{n+1,k_{n+1}}} + \mathbf{1}_{E_{n+1,k_{n+1}+\theta}}\,.$$

Then

$$\begin{aligned}\left\|(Tf_n)(t) - (Tf_{n+1})(t)\right\| &= 2^n \left\|T\big(\mathbf{1}_{E_{n,k_n}} - 2\mathbf{1}_{E_{n+1,k_{n+1}}}\big)\right\| \\ &= 2^n \left\|T\big(\mathbf{1}_{E_{n+1,k_{n+1}+\theta}} - \mathbf{1}_{E_{n+1,k_{n+1}}}\big)\right\| \ge 2^n \frac{\delta}{2^{n+1}} = \frac{\delta}{2}\,.\end{aligned}$$

Thus, the sequence $(f_n(t))$ is divergent. □

Proof of Theorem 8.17. If T is narrow then so is ST, by Proposition 1.8.

For the other direction, suppose that ST is narrow. Assume on the contrary that T is not narrow. By Theorem 7.59, T is not somewhat narrow, and thus there exist $A \in \Sigma^+$ and $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for each sign x with $\operatorname{supp} x \subseteq A$. With no loss of generality we may and do assume that $A = [0, 1]$. Indeed, by the Carathéodory theorem, there exists a linear isometry $J : L_1 \to L_1(A)$ sending signs to signs. Then the operator $T_1 = TJ \in \mathcal{L}(L_1, X)$ satisfies $\|T_1 x\| \geq \delta\|x\|$ for every sign x, and $ST_1 = STJ$ is narrow.

Set $G = S(\overline{T\mathcal{P}})$. By Lemma 8.20, there exists a standard dyadic martingale (f_n) with (STf_n) converging a.e. to a function g valued in G. By Lemma 8.19, (Tf_n) converges a.e. which contradicts Lemma 8.21. □

A class of Banach spaces X such that L_1 sign-embeds in X if and only if L_1 isomorphically embeds in X

Let $\mathfrak{G}$ be the minimal class of separable Banach spaces such that

(a) $L_1 \in \mathfrak{G}$;

(b) If $Y \in \mathfrak{G}$ and X G_δ-embeds in Y, then $X \in \mathfrak{G}$.

By minimality, $\mathfrak{G} = \bigcup_{n=1}^\infty \mathfrak{G}_n$, where $\mathfrak{G}_n$ is the class of all separable Banach spaces X such that there are Banach spaces $X_1, \dots, X_n$ and G_δ-embeddings $T_i \in \mathcal{L}(X_i, X_{i+1})$ for $i = 0, \dots, n$, where $X_0 = X$ and $X_{n+1} = L_1$.

As pointed out by Ghoussoub and Rosenthal in [44], the class $\mathfrak{G}$ contains all separable dual spaces. Indeed, by a result of Bourgain and Rosenthal [20], all separable duals semi-embed in ℓ_2 which, in turn, isomorphically embeds in L_1. Thus, all separable duals semi-embed (and hence, G_δ-embed) in L_1.

Theorem 8.22. *Suppose that a Banach space $X \in \mathfrak{G}$ contains no subspace isomorphic to L_1. Then every operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Proof. Fix any $T \in \mathcal{L}(L_1, X)$. Let $X = X_0, X_1, \dots, X_n, X_{n+1} = L_1$ be Banach spaces, so that there exist G_δ-embeddings $T_i \in \mathcal{L}(X_i, X_{i+1})$ for $i = 0, \dots, n$. Consider the operator $\tilde{T} = T_{n+1}T_n \dots T_1 T : L_1 \to L_1$. Since T is L_1-singular, so is $\tilde{T}$. By Theorem 7.30, $\tilde{T}$ is narrow. Using Theorem 8.17 $n + 1$ times, we obtain that operators $T_n \dots T_1 T, T_{n-1} \dots T_1 T, \ \dots \ , T_1 T$ and T are narrow. □

The following immediate consequence of Theorems 8.22 and 8.4 gives a partial positive answer to Rosenthal's Problem 8.8.

Corollary 8.23. *Suppose $X \in \mathfrak{G}$. If L_1 sign-embeds in X then L_1 isomorphically embeds in X.*

As noted above, in general, Problem 8.8 has a negative answer, which is given by a counterexample of Talagrand presented in the next section.

8.5 Sign-embeddability of L_1 does not imply isomorphic embeddability

Talagrand in [138] solved in the negative the three-space problem for isomorphic embeddings in L_1. His remarkable example also shows that L_1 sign-embeddability is distinct from the isomorphic embeddability.

Theorem 8.24 (Talagrand [138]). *There exists a subspace Z of L_1 such that neither Z nor L_1/Z contains an isomorph of L_1.*

Corollary 8.25. *There exists a Banach space X such that L_1 sign-embeds in X but L_1 does not embed isomorphically in X.*

Proof. Let X be the quotient space L_1/Z where Z is the subspace of L_1 from Theorem 8.24. We show that X has the desired properties. By Theorem 8.24, X contains no subspace isomorphic to L_1. By Corollary 2.23, X is not rich, and hence, the quotient map $T : L_1 \to X$ is not narrow. By Theorem 7.64, T is not somewhat narrow, that is, there are $\delta > 0$ and $A \in \Sigma^+$ such that $\|Tx\| \geq \delta \|x\|$ for each sign x with $\operatorname{supp} x \subseteq A$. Let $J : L_1 \to L_1(A)$ be a linear isometry sending signs to signs (it exists by the Carathéodory theorem). Then the operator $T_1 = TJ \in \mathcal{L}(L_1, X)$ is a generalized sign-embedding. By Proposition 8.6, L_1 sign-embeds in X. □

Outline of the proof of Theorem 8.24

The proof is rather long and it requires several auxiliary results, so we start with an outline of the whole proof postponing the proofs of intermediate claims and propositions to the end of the section. In our proof we follow [138].

The main idea of the proof is to apply Rosenthal's characterization of subspaces X of L_1 for which the quotient map from L_1 onto L_1/X does not fix a copy of L_1.

Corollary 8.26. ([128, Theorem 2.1]) *For a subspace X of L_1 the following are equivalent:*

(8.26.1) *The quotient map $L_1 \to L_1/X$ does not fix a copy of L_1.*

(8.26.2) *For each $\delta > 0$, each $A \in \Sigma^+$, each atomless sub-σ-algebra Σ' of $\Sigma(A)$ there exist $x \in L_1(A, \Sigma')$ and $y \in X$ such that $\|x\|_1 \geq \frac{1}{4}$ and $\|x-y\|_1 \leq \delta$.*

Corollary 8.26 is a reformulation of Theorem 7.80.

Our goal is to construct a family of functions satisfying (8.26.2) in such a way that if X is the span of these functions, then, in addition, neither X nor L_1/X contain L_1. We will achieve this by choosing functions x for which $\|x\|_\infty$ and $\|x\|_1$ are not of the same order, and which we will control in measure. The idea of employing convergence in measure in this context goes back to Roberts [120].

For each $m = 0, 1, \ldots$, we denote by $\mathcal{F}_m$ the collection $\{I_m^k : 1 \le k \le 2^m\}$ of dyadic intervals of length 2^{-m} and by Σ_m the algebra that they generate. For any $x \in L_1$, $\mathbb{E}^m x$ will denote the conditional expectation of x with respect to Σ_m, so for $I \in \mathcal{F}_m$, the constant value of $\mathbb{E}^m x$ on I is $2^m \int_I x \, \mathrm{d}\mu$.

We now will describe the families D_n that we want to construct.

Let S_{2n-1} be the set of all strictly increasing functions $\sigma : \{0, \ldots, 2n-1\} \to \mathbb{N}$. Consider a collection $B_0, \ldots, B_{2n-1}$ in Σ and $\sigma \in S_{2n-1}$ with the following properties:

$$B_i \text{ is } \Sigma_{\sigma(i)}\text{-measurable;} \tag{8.8}$$

$$\mathbb{E}^{\sigma(i)}(\mathbf{1}_{B_{i+1}}) = \frac{1}{2}\mathbf{1}_{B_i}, \text{ for } 0 \le i < 2n-1. \tag{8.9}$$

This latter condition means that $B_{i+1} \subseteq B_i$, and that for each $I \in \mathcal{F}_{\sigma(i)}$ with $I \subseteq B_i$, we have $\mu(I \cap B_{i+1}) = \frac{1}{2}\mu(I)$.

Let

$$y = \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} (-1)^i 2^i \mathbf{1}_{B_i} \,. \tag{8.10}$$

If we denote $B_{2n} = \emptyset$, then

$$y = \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} a_i \mathbf{1}_{B_i \setminus B_{i+1}} \,, \tag{8.11}$$

where $a_i = \sum_{0 \le l \le i} (-1)^l 2^l = ((-1)^i 2^{i+1} + 1)/3$.

We define D_n to be the set of all $y \in L_1$ given by (8.10) for all possible choices of $B_0, \ldots, B_{2n-1}$ that satisfy (8.8) and (8.9) with respect to some $\sigma \in S_{2n-1}$, and $\mu(B_0) \ge 2^{-n}$. Observe that

$$y \in D_n \implies \|y\|_\infty \le 2^{3n} \,. \tag{8.12}$$

We also have

$$y \in D_n \implies \|y\|_1 \ge \frac{1}{3} \,. \tag{8.13}$$

Indeed, since $\mu(B_i) = 2^{-i}\mu(B_0)$,

$$\begin{aligned}
\|y\|_1 &= \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} |a_i| \mu(B_i \setminus B_{i+1}) \\
&\ge \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} \frac{(-1)^i 2^{i+1} + 1}{3} 2^{-i-1} \mu(B_0) \\
&= \frac{1}{6n} \Big(\sum_{i=0}^{2n-1} (1 + (-1)^i 2^{-i-1}) \Big) \ge \frac{1}{3},
\end{aligned}$$

since $\sum_{i=0}^{2n-1} (-1)^i 2^{-i-1} \ge 0$.

First we claim that for every n, the set D_n satisfies a condition similar to (8.26.2).

Proposition 8.27. *For each $n \in \mathbb{N}$, each set $A \in \Sigma$ with $\mu(A) > 2^{-n}$, each atomless sub-σ-algebra Σ' of $\Sigma(A)$ and every η with $0 < \eta < \frac{1}{12}$, there exist $x \in L_1(A, \Sigma')$ and $y \in D_n$, such that $\|x\|_1 \geq \frac{1}{4}$ and $\|x - y\|_1 \leq \eta$.*

We denote by absconv D_n the absolute convex hull of D_n, i.e. the set of elements $\sum_{l \in L} c_l y_l$, where L is a finite set, $y_l \in D_n$ and $\sum_{l \in L} |c_l| \leq 1$.

Our principal tool is the following result about the control in measure of elements of the sets absconv D_n.

Theorem 8.28. *For any $\varepsilon > 0$, there exists $n > 0$ such that $\mu(\{|y| \geq \varepsilon\}) \leq \varepsilon$ for every $y \in$ absconv D_n.*

Our task is to construct small perturbations of the functions in D_n in such a way that their span will be isomorphic to ℓ_1. To do this, we construct recursively a sequence of numbers $q(l) \in \mathbb{N}$ and a sequence of disjoint measurable sets $(B_l)_l$ satisfying the following conditions:

$$0 < \mu(B_l) \leq 2^{-3q(l-1)-2l-2}, \tag{8.14}$$

$$\mu(\{|x| \geq 2^{-2l}\}) \leq 2^{-l-1} \min_{i \leq l} \mu(B_i), \quad \forall x \in \text{absconv}\, D_{q(l)}. \tag{8.15}$$

We start the construction with $B_1 = [0, 2^{-4}]$.

At each stage of the construction, using Theorem 8.28, we choose $q(l)$ so that (8.15) holds. Then we choose B_{l+1} satisfying (8.14).

Now we fix a one-to-one map $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and define $A(n,k) = B_{\varphi(n,k)}$.

Let $(x_{n,k})_{k \geq 1}$ be a sequence in $C_n \stackrel{\text{def}}{=} D_{q(n)}$, and

$$y_{n,k} = x_{n,k} + 2^{-n} \frac{1}{\mu(A(n,k))} \mathbf{1}_{A(n,k)} \,. \tag{8.16}$$

The next proposition asserts that $(y_{n,k})_{k \geq 1}$ is the sequence of small perturbations that we are looking for.

Proposition 8.29. *For each $n \in \mathbb{N}$ and for each sequence (a_k) of numbers with all but finitely many equal to zero, we have*

$$\|y_{n,k}\|_1 \leq 1 + 2^{-n} \leq 2, \tag{8.17}$$

$$\|y_{n,k} - x_{n,k}\|_1 \leq 2^{-n}, \tag{8.18}$$

$$2^{-n-2} \sum_{k=1}^{\infty} |a_k| \leq \Big\| \sum_{k=1}^{\infty} a_k y_{n,k} \Big\|_1 \leq 2 \sum_{k=1}^{\infty} |a_k| \,. \tag{8.19}$$

We will use our main tool, Theorem 8.28, to show that the following result is important for our further construction.

Proposition 8.30. *For any sequence $u_n \in 2^{n+2}\widetilde{C}_n$, where $\widetilde{C}_n$ is the norm closure of* conv C_n*, the series $\sum_{n\geq 1} u_n$ converges in measure, and if its sum belongs to L_1 then*

$$\sum_{n=1}^{\infty} \|u_n\|_1 \leq \Big\|\sum_{n=1}^{\infty} u_n\Big\|_1 + 6\,. \tag{8.20}$$

Next, for each $n \in \mathbb{N}$, we define H_n to be the closed linear span of the sequence $(y_{n,k})_{k\geq 1}$. By (8.19), for each $n \in \mathbb{N}$, the space H_n is isomorphic to ℓ_1. Moreover we obtain that the span of the spaces H_n is isomorphic to their ℓ_1-sum. More precisely, we have the following.

Proposition 8.31. *For each sequence $h_n \in H_n$ that is eventually zero, we have*

$$\frac{1}{15}\sum_{n=1}^{\infty} \|h_n\|_1 \leq \Big\|\sum_{n=1}^{\infty} h_n\Big\|_1 \leq \sum_{n=1}^{\infty} \|h_n\|_1\,.$$

Now we are ready to define the space X which is the objective of our construction .

$$X \stackrel{\text{def}}{=} \overline{\text{span}}\Big(\bigcup_{n=1}^{\infty} H_n\Big)\,.$$

By Propositions 8.29 and 8.31, the space X is isomorphic to an ℓ_1-sum of spaces isomorphic to ℓ_1, and thus does not contain L_1. It follows from the construction and Proposition 8.27 and Corollary 8.26, that the quotient map T from L_1 onto L^1/X does not fix a copy of L_1. It remains to be shown that L^1/X does not contain a copy of L_1.

For this we will use the following result in the spirit of [44].

Proposition 8.32. *Let T be an operator from a Banach space Y onto a Banach space Z. For $z \in Z$ with $\|z\| < 1$, let $U_z = T^{-1}(\{z\}) \cap B_Y$. Suppose that there exists a constant $K > 0$ so that for every sequence $(y_n)_{n=1}^{\infty} \in U_z$, there exists a point $\varphi((y_n)_{n=1}^{\infty}) \in Y$ with the following properties:*

$$\varphi\big((x_n)_{n=1}^{\infty}\big) = \varphi\big((y_n)_{n=1}^{\infty}\big) \text{ if } x_n = y_n \text{ for } n \text{ large enough,} \tag{8.21}$$

$$\big\|\varphi\big((y_n)_{n=1}^{\infty}\big)\big\| \leq K, \tag{8.22}$$

$$T\big(\varphi\big((y_n)_{n=1}^{\infty}\big)\big) = z, \tag{8.23}$$

$$\varphi\Big(\Big(\frac{y_n + y_n'}{2}\Big)_{n\geq 1}\Big) = \frac{1}{2}\Big[\varphi\big((y_n)_{n=1}^{\infty}\big) + \varphi\big((y_n')_{n=1}^{\infty}\big)\Big], \quad \forall y_n, y_n' \in U_z. \tag{8.24}$$

If, in addition, there exists an isomorphism V from L_1 into Z, then there exists $S \in \mathcal{L}(L_1, Y)$ so that $\|S\| \leq K$ and $V = T \circ S$, and, in particular, T fixes a copy of L_1.

To finish the proof of Theorem 8.24, we denote by L_0 the space of all measurable functions on $[0,1]$ considered with the topology of convergence in measure. Let $M_n = 2^n \widetilde{C}_n$, the absolute convex hull of $2^n C_n = 2^n D_{q(n)}$, and define the set $G \subset L_0$ as follows:

$$G \stackrel{\text{def}}{=} \{x_1 + \sum_{n=1}^{\infty} \alpha_n y_n : x_1 \in L_1, \|x_1\|_1 \le 1, y_n \in M_n, \sum_{n=1}^{\infty} |\alpha_n| \le 1\}.$$

We observe that by Proposition 8.30, the series $\sum_{n=1}^{\infty} \alpha_n y_n$ converges in measure and that G is bounded in L_0. Let N be the Minkowski functional of the set G given by $N(z) = \inf\{a > 0 : z \in aG\}$, if this set is nonempty, and $N(z) = \infty$ otherwise. We set $E = \{z \in L^0 : N(z) < \infty\}$, and provide E with norm N. Since G is bounded in L_0, E is a Banach space, and the canonical map j from L_1 to E is one-to-one. Since the unit ball of L_1 is closed in L_0, j is a semi-embedding. Moreover E has the following properties.

Proposition 8.33.

(a) *E does not contain L_1.*

(b) *Let W be the closure of $j(X)$ in E. Then W is isomorphic to ℓ_1.*

(c) *L_1/X is isomorphic to E/W.*

Thus to prove that L_1/X does not contain a copy of L_1, it is enough to prove that E/W does not contain a copy of L_1. This will follow from Proposition 8.32.

Indeed, by Proposition 8.33(b), W is isomorphic to ℓ_1, which is a dual space. Denote by τ the weak* topology on W induced by an isomorphism with ℓ_1, and fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $Z = E/W$ and V be the quotient map from E onto Z. For a sequence $(y_n)_{n=1}^{\infty}$ in $U_z = \{y \in E : \|y\| \le 1, V(y) = z\}$, we set $\varphi((y_n)_{n=1}^{\infty}) = y + \lim_{n \in \mathcal{U}}(y_n - y)$, where y is any point of $V^{-1}(z)$ and the limit is taken in the τ topology in W. It is clear that (8.21)–(8.24) are satisfied. Since, by Proposition 8.33(a), E does not contain a copy of L_1, Proposition 8.32 implies that Z cannot contain a copy of L_1, which ends the proof of Theorem 8.24.

Proofs

We now present the proofs of all results used in the proof of Theorem 8.24.

Proof of Proposition 8.27

This proof is a standard technical exercise. We need the following lemma.

Lemma 8.34. *For all $\varepsilon > 0$, $k \in \mathbb{N}$, $A \in \Sigma$, $B \in \Sigma_k$ with $\mu(A \triangle B) < \varepsilon$, and all $A' \subset A$ so that $\mu(A' \cap I) = \frac{1}{2}\mu(A \cap I)$ for all $I \in \mathcal{F}_k$, there exist $m > k$ and $B' \in \Sigma_m$, so that $B' \subset B$, $\mu(A' \triangle B') < 4\varepsilon$ and $\mathbb{E}^k(\mathbf{1}_{B'}) = \frac{1}{2}\mathbf{1}_B$.*

We first prove the proposition using the lemma, and then we prove the lemma.

Let n, A, Σ' and η be as in the assumptions of the proposition. Let $0 < \varepsilon < \mu(A) - 2^{-n}$. By induction over i, $0 \le i \le 2n-1$, we will construct numbers $\sigma(i)$ and sets $A_i \in \Sigma'$, $B_i \in \Sigma_{\sigma(i)}$ so that (8.8) and (8.9) hold and

$$A_0 = A,\ \ A_{i+1} \subset A_i,\ \ \mu(A_{i+1}) = \frac{1}{2}\mu(A_i),\ \ \mu(A_i \triangle B_i) \le 4^i \varepsilon.$$

We start the induction with $\sigma(0) \in \mathbb{N}$ and $B_0 \in \Sigma_{\sigma(0)}$ so that $\mu(A \triangle B_0) \le \varepsilon$. When $A_i, B_i, \sigma(i)$ have been constructed, since Σ' is atomless, by Lyapunov's convexity theorem, there exists $A_{i+1} \in \Sigma'$ so that $A_{i+1} \subset A_i$ and

$$\int_{A_{i+1}} \mathbf{1}_I \, d\mu = \mu(A_{i+1} \cap I) = \frac{1}{2}\mu(A_i \cap I),$$

for all $I \in \mathcal{F}_{\sigma(i)}$. In particular, $\mu(A_{i+1}) = \frac{1}{2}\mu(A_i)$. By Lemma 8.34, there exist $\sigma(i+1) > \sigma(i)$ and $B_{i+1} \in \Sigma_{\sigma(i+1)}$ so that $\mu(A_{i+1} \triangle B_{i+1}) \le 4\mu(A_i \triangle B_i)$ and $\mathbb{E}^{\sigma(i)}(\mathbf{1}_{B_{i+1}}) = \frac{1}{2}\mathbf{1}_{B_i}$. This completes the construction.

The function

$$x = \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} (-1)^i 2^i \mathbf{1}_{A_i}$$

is Σ'-measurable and is supported on $A_0 = A$. Let

$$y = \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} (-1)^i 2^i \mathbf{1}_{B_i} \ .$$

Since $\mu(B_0) \ge \mu(A) - \varepsilon \ge 2^{-n}$ we have $y \in D_n$. Moreover

$$\|x - y\|_1 = \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} 2^i \|\mathbf{1}_{B_i} - \mathbf{1}_{A_i}\|_1 \le \frac{1}{2n\mu(B_0)} \sum_{i=0}^{2n-1} 2^i 4^i \varepsilon \le \frac{8^{2n+1}}{2n\mu(B_0)}\varepsilon \ .$$

Since n is fixed, $\|x - y\|_1$ can be made smaller than η. Since $\eta < \frac{1}{12}$, by (8.13), we also get that $\|x\|_1 \ge \frac{1}{3} - \eta \ge \frac{1}{4}$.

Proof of Lemma 8.34. Let $A_1 = A' \cap B$. Since $A' \setminus A_1 = A' \setminus B \subset A \setminus B$, we get $\mu(A' \triangle A_1) = \mu(A' \setminus A_1) \le \varepsilon$.

Let $A_2 \in \Sigma$, $A_1 \subset A_2 \subset B$, with $A_2 \cap A = A_1$, be such that for all $I \in \mathcal{F}_k$, $I \subset B$, we have $A \cap A_2 \cap I = A_1 \cap I = A' \cap I$ and $\mu(A_2 \cap I) = \frac{1}{2}\mu(I)$. Then $\mu(A_2 \triangle A_1) = \mu(A_2 \setminus A_1) \le \mu(B \setminus A) \le \varepsilon$.

There exists $m > k$ and $B_1 \in \Sigma_m$ so that $\mu(B_1 \triangle A_2) \le 2^{-k}\varepsilon$.

For every $I \in \mathcal{F}_k$, $I \subset B$, we have $|\mu(B_1 \cap I) - 2^{-k-1}| = |\mu(B_1 \cap I) - \mu(A_2 \cap I)| \le \mu(B_1 \triangle A_2) \le 2^{-k}\varepsilon$. Moreover, there exists $B' \in \Sigma_m$, $B' \subset B$, such that for all $I \in \mathcal{F}_k$, $I \subset B$, we have $\mu(B' \cap I) = 2^{-k} = \frac{1}{2}\mu(B' \cap I)$ and $\mu((B' \cap I)\triangle(B_1 \cap I)) \le 2^{-k}\varepsilon$. Hence $\mu(B'\triangle B_1) \le \varepsilon$.

Combining all of the above, we get

$$\mu(A'\triangle B') \le \mu(A'\triangle A_1) + \mu(A_1\triangle A_2) + \mu(A_2\triangle B_1) + \mu(B_1\triangle B') \le 4\varepsilon. \qquad \square$$

Proof of Theorem 8.28

The idea of this proof is to first replace sets D_n with sets $F_n \subset L_1$, whose convex hulls contain D_n but which consist of elements of a simpler form than those of D_n, and then to prove the result for these larger sets.

Given $t \in [0,1]$ and $m \in \mathbb{N}$, we denote by $I(t,m)$ the unique dyadic interval from $\mathcal{F}_m$ containing t. Given $\sigma \in S_{2n-1}$, i.e. an increasing sequence of natural numbers $\sigma(1) < \sigma(2) < \ldots < \sigma(2n-1)$, we set

$$x(t,\sigma) = \frac{1}{2n}\sum_{i=0}^{2n-1}(-1)^i 2^{\sigma(i)}\mathbf{1}_{I(t,\sigma(i))}\,, \tag{8.25}$$

$$F_n = \{x(t,\sigma) \in L_1 : t \in [0,1],\ \sigma \in S_{2n-1}\}\,.$$

Since $\mu(I(t,\sigma(i))) = 2^{-\sigma(i)}$, we have $\|x(t,\sigma)\|_1 \le 1$.

For each $t \in [0,1]$ and $\sigma \in S_{2n-1}$ we set $C = I(t,\sigma(2n-1))$. Since $\mathbb{E}^{\sigma(i)}\mathbf{1}_C = 2^{\sigma(i)-\sigma(2n-1)}\mathbf{1}_{I(t,\sigma(i))}$, we have

$$x(t,\sigma) = \frac{1}{2n}\sum_{i=0}^{2n-1}\frac{(-1)^i}{\mu(C)}2^{\sigma(i)}\mathbb{E}^{\sigma(i)}\mathbf{1}_C(t)\,. \tag{8.26}$$

First we prove that the sets F_n are large enough.

Lemma 8.35. *For every $n = 1,2,\ldots$, the convex hull of F_n contains D_n.*

Proof. Let $y \in D_n$ be given by (8.10). Then (8.9) implies that for every $0 \le i \le 2n-1$ we have

$$\mu(B_{2n-1})^{-1}\mathbb{E}^{\sigma(i)}\mathbf{1}_{B_{2n-1}} = 2^{2n-1}\mu(B_0)^{-1}\mathbf{1}_{B_{2n-1}} = 2^i\mu(B_0)^{-1}\mathbf{1}_{B_i}\,,$$

and hence

$$y = \frac{1}{2n}\sum_{i=0}^{2n-1}\frac{(-1)^i}{\mu(B_{2n-1})}\mathbb{E}^{\sigma(i)}\mathbf{1}_{B_{2n-1}}\,. \tag{8.27}$$

By (8.8), $B_{2n-1} = \bigsqcup_{l\in L} C_l$, where L is finite and $C_l \in \mathcal{F}_{\sigma(2n-1)}$. Thus, by (8.27),

$$y = \frac{1}{\operatorname{card} L}\left(\frac{1}{2n}\sum_{i=0}^{2n-1}\frac{(-1)^i}{\mu(C_l)}\mathbb{E}^{\sigma(i)}\mathbf{1}_{C_l}\right).$$

By (8.26), y belongs to the convex hull of F_n. $\square$

Now we need more notation. For each $n \in \mathbb{N}$, $0 \le j,k < n$, $t \in [0,1]$ and $\sigma \in S_{2n-1}$, we set

$$x_j(t,\sigma) = 2^{\sigma(2j)}\mathbf{1}_{I(t,\sigma(2j))} - 2^{\sigma(2j+1)}\mathbf{1}_{I(t,\sigma(2j+1))} .$$

and

$$x(t,\sigma,k) = \frac{1}{2n}\sum_{j=0}^{k-1} x_j(t,\sigma) = \frac{1}{2n}\sum_{i=0}^{2k-1}(-1)^i 2^{\sigma(i)}\mathbf{1}_{I(t,\sigma(i))} .$$

Thus, $x(t,\sigma,n) = x(t,\sigma)$ and $\|x(t,\sigma,k)\|_1 \le k/n$, for all values of the parameters.

Lemma 8.36. *Let L be a finite set and $n,r \in \mathbb{N}$. For each $l \in L$, let $t_l \in [0,1]$ $\sigma_l \in S_{2n-1}$, $k_l \in \{1,\dots,n\}$ and $\alpha_l \in \mathbb{R}$ with $\sum_{l\in L}|\alpha_l| \le 1$. Consider $x = \sum_{l\in L}\alpha_l x(t_l,\sigma_l,k_l)$. Assume that $\|x\| \ge 2^{-r}$ and $n \ge 3(r+1)2^{3r+7}$. Then there exist $A \in \Sigma$ with $\mu(A) \le 2^{-2r}$, and numbers $m_l \in \{1,\dots,k_l\}$ such that*

$$\frac{1}{n}\sum_{l\in L}\alpha_l m_l \le \frac{1}{n}\sum_{l\in L}\alpha_l k_l - 2^{-r-1}, \tag{8.28}$$

$$x'(t) = x(t), \quad \textit{for all } t \in [0,1]\setminus A, \;\; \textit{where } x' = \sum_{l\in L}\alpha_l x(t_l,\sigma_l,m_l) . \tag{8.29}$$

Proof. The main ingredient of the proof is the following estimate.

Given y with $\|y\|_\infty \le 1$, $m \in \{0,1,\dots\}$, $\alpha,\beta \in \mathbb{R}$ with $\alpha > \beta$, we set

$$A(y,m,\alpha,\beta) =$$
$$\big\{t : (\exists\sigma \in S_m)(\forall i \in \{0,\dots,m-1\})\big((\mathbb{E}^{\sigma(2i)}y)(t) \ge \alpha\big)\&\big((\mathbb{E}^{\sigma(2i+1)}y)(t) \le \beta\big)\big\}$$

Then

$$\mu\big(A(y,m,\alpha,\beta)\big) \le \Big(\frac{1-\alpha}{1-\beta}\Big)^m . \tag{8.30}$$

This follows from Doob's inequality [103, p. 27] applied to the martingale $(\mathbb{E}^n(1-y))_n$ (or it can be proved directly by induction on m).

Now consider $y = \operatorname{sign} x$. We have $\|y\|_\infty \le 1$ and

$$2^{-r} \le \|x\|_1 = \int_{[0,1]} xy\,\mathrm{d}\mu = \frac{1}{2n}\sum_{l\in L}\alpha_l\sum_{j=0}^{k_l-1}\int_{[0,1]} yx_j(t_l,\sigma_l)\,\mathrm{d}\mu . \tag{8.31}$$

Let $m = 3(r+1)2^{r+3}$, and for each $l \in L$, let m_l be the largest integer $\le k_l$ so that

$$\operatorname{card}\Big\{i \in \{0,\dots,k_l\} : \int_{[0,1]} yx_i(t_l,\sigma_l)\,\mathrm{d}\mu \ge 2^{-r-1}\Big\} \le 2^{r+3}m .$$

Let $L' = \{l \in L : m_l < k_l\}$. Then for every $l \in L'$ we have

$$\operatorname{card}\Big\{i \in \{0,\dots,k_l\} : \int_{[0,1]} y x_i(t_l,\sigma_l)\,\mathrm{d}\mu \geq 2^{-r-1}\Big\} = 2^{r+3} m\,. \tag{8.32}$$

Since $\int_{[0,1]} y 2^m \mathbf{1}_{I(t,m)}\,\mathrm{d}\mu = (\mathbb{E}^m y)(t)$, we have that

$$\int_{[0,1]} y x_j(t_l,\sigma_l)\,\mathrm{d}\mu = (\mathbb{E}^{\sigma_l(2j)} y)(t_l) - (\mathbb{E}^{\sigma_l(2j+1)} y)(t_l)\,.$$

Hence, if $\int_{[0,1]} y x_j(t_l,\sigma_l)\,\mathrm{d}\mu \geq 2^{-r-1}$, then there exists $s \in \mathbb{Z}$ with $-2^{r+2} \leq s < 2^{r+2}$ such that

$$(\mathbb{E}^{\sigma_l(2j)} y)(t_l) \geq (s+1)\,2^{-r-2}, \quad (\mathbb{E}^{\sigma_l(2j+1)} y)(t_l) \leq s\,2^{-r-2}\,. \tag{8.33}$$

By (8.32), for every $l \in L'$ there exists $s \in \mathbb{Z}$ with $-2^{r+2} \leq s < 2^{r+2}$ such that (8.33) holds for, at least, m indices $j \leq m_l - 1$. Hence,

$$I\big(t_l, \sigma_l(2m_l-1)\big) \subseteq A_s \overset{\text{def}}{=} A\big(y, m, (s+1)\,2^{-r-2}, s\,2^{-r-2}\big)$$

By (8.30),

$$\mu(A_s) \leq \Big(\frac{1-(s+1)\,2^{-r-2}}{1-s\,2^{-r-3}}\Big)^m \leq \big(1-2^{-r-3}\big)^m \leq e^{-m2^{-r-3}} \leq 2^{-3r-3}\,.$$

Thus, for $A = \bigcup_{-2^{r+2} \leq s < 2^{r+2}} A_s$ we have $\mu(A) \leq 2^{-2r}$. Note that if $j \geq 2m_l$, then $I(t,\sigma_l(j)) \subseteq A$, which implies (8.29).

Since for each $l \in L$

$$\Big|\int_{[0,1]} y x_j(t_l,\sigma_l)\,\mathrm{d}\mu\Big| \leq \|y\|_\infty \|x_j(t_l,\sigma_l)\|_1 \leq 2\,,$$

we have

$$\Big|\int_{[0,1]} y x(t,\sigma_l,m_l)\,\mathrm{d}\mu\Big| \leq \frac{1}{2n}\big(2^{r+4}m + n2^{-r-1}\big) \leq 2^{r+3}\frac{m}{n} + 2^{-r-2} \leq 2^{-r-1}\,.$$

This implies that $\sum_{l\in L} \alpha_l \int_{[0,1]} y x_j(t,\sigma_l)\,\mathrm{d}\mu \leq 2^{-r-1}$. On the other hand, by (8.31), we get

$$\begin{aligned} 2^{-r-1} &\leq \sum_{l\in L} \alpha_l \int_{[0,1]} y\big(x(t_l,\sigma_l,k_l) - x(t_l,\sigma_l,m_l)\big)d\mu \\ &\leq \sum_{l\in L} |\alpha_l| \big\|x(t_l,\sigma_l,k_l) - x(t_l,\sigma_l,m_l)\big\|_1 \end{aligned} \tag{8.34}$$

Finally, since $\|x(t_l,\sigma_l,k_l) - x(t_l,\sigma_l,m_l)\|_1 \leq (k_l - m_l)/n$, (8.34) implies (8.28). □

Lemma 8.37. *For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, such that $\mu(\{|x| \geq \varepsilon\}) \leq \varepsilon$ for each $x \in \operatorname{conv} F_n$.*

Proof. Let $r, n \in \mathbb{N}$, so that $2^{2-r/2} < \varepsilon$ and $n > 3(r+1)\,2^{3r+7}$. An arbitrary element of $\operatorname{conv} F_n$ has the following form:

$$x = \sum_{l \in L} \alpha_l x(t_l, \sigma_l, k_l),$$

where L is a finite set, and for each $l \in L$, $t_l \in [0,1]$, $\sigma_l \in S_n$, and $(\alpha_l)_{l \in L} \subset \mathbb{R}$ satisfy $\sum_{l \in L} |\alpha_l| \leq 1$.

Recursively on $\nu = 0, 1, \ldots$, we construct natural numbers $(k_l^\nu)_{l \in L}$ with $0 \leq k_l^\nu \leq n$ and sets $A_\nu \in \Sigma'$ with the following properties:

$$\frac{1}{n} \sum_{l \in L} |\alpha_l| k_l^\nu \leq \frac{1}{n} \sum_{l \in L} |\alpha_l| k_l^{\nu-1} - 2^{-r-1}\,, \tag{8.35}$$

$\mu(A_\nu) \leq 2^{-2r}$ and $x^{(\nu)}(t) = x(t)$ for all $t \in [0,1] \setminus A'_\nu$, where $A'_\nu = \bigcup_{i<\nu} A_i$ and $x^{(\nu)} = \sum_{l \in L} \alpha_l x(t_l, \sigma_l, k_l^\nu)$.

To start the construction, let $k_l^0 = n$ for all $l \in L$ and $A_0 = \emptyset$. The induction step is done by applying Lemma 8.36. The construction stops at the first value of ν for which $\|x^{(\nu)}\| \leq 2^{-r}$. Thus, $\mu(\{|x^{(\nu)}| \geq 2^{-r/2}\}) \leq 2^{-r/2}$. By (8.35), we have

$$0 \leq \frac{1}{n} \sum_{l \in L} |\alpha_l| k_l^\nu \leq 1 - \nu 2^{-r-1}\,,$$

and thus, $\nu \leq 2^{r+1}$. Hence, $\mu(A'_\nu) \leq \nu 2^{-2r} \leq 2^{-r+1}$. Since x and $x^{(\nu)}$ coincide outside A'_ν, we obtain

$$\mu\big(\{|x| \geq \varepsilon\}\big) \leq \mu\big(\{|x| \geq 2^{-r/2}\}\big) \leq 2^{-r/2} + 2^{-r+1} \leq 2^{2-r/2} < \varepsilon\,. \qquad \square$$

By Lemma 8.35, Theorem 8.28 follows from Lemma 8.37.

Proof of Proposition 8.29

Observe that (8.17) and (8.18) easily follow from (8.16). The upper estimate in (8.19) follows immediately from (8.17), so we just need to prove the lower estimate in (8.19). We do this in the following lemma.

Lemma 8.38. *Let $(a_{n,k})_{n,k=1}^\infty$ be a sequence of reals with $\sum_{n=1}^\infty 2^{-n} \sum_{k=1}^\infty |a_{n,k}| < \infty$. Then the series $\sum_{n=1}^\infty (\sum_{k=1}^\infty a_{n,k} y_{n,k})$ converges in measure. Denoting for simplicity its sum by $\sum_{n,k=1}^\infty a_{n,k} y_{n,k}$, we have*

$$\sum_{n,k=1}^\infty 2^{-n} |a_{n,k}| \leq 4 \Big\| \sum_{n,k=1}^\infty a_{n,k} y_{n,k} \Big\|_1\,. \tag{8.36}$$

Proof of Lemma 8.38. For $n \in \mathbb{N}$, let $b_n = \sum_{k=1}^{\infty} |a_{n,k}|$. Thus $\sum_{n=1}^{\infty} 2^{-n} b_n < \infty$. Let $h_n = \sum_{k=1}^{\infty} a_{n,k} x_{n,k}$. Then, for all $n \in \mathbb{N}$, $h_n \in b_n \widetilde{C}_n$, where $\widetilde{C}_n$ is the norm closure of $\operatorname{conv} C_n$. By (8.15), the series $\sum_{n=1}^{\infty} (\sum_{k=1}^{\infty} a_{n,k} y_{n,k})$ converges in measure. By (8.12), we have that $\|h_n\|_\infty \le b_n 2^{3q(n)}$ and hence, by (8.14), for all $l > n$,

$$\int_{B_l} |h_n| \, d\mu \le b_n 2^{3q(n)} \mu(B_l) \le b_n 2^{-2l-2} \le b_n 2^{-l-n-2} . \tag{8.37}$$

Let $A_n = \{|h_n| \ge b_n 2^{-2n}\}$. By (8.15), $\mu(A_n) \le 2^{-n-1} \min_{i \le n} \mu(B_i)$. Let $A'_l = \bigcup_{n \ge l} A_n$. Since $\mu(A_n) \le 2^{-n-1} \mu(B_l)$, for all $n \ge l$, we have that $\mu(A'_l) \le 2^{-l} \mu(B_l) \le \mu(B_l)/2$. Let $B'_l = B_l \setminus A'_l$. Then, for all $l \le n$, we have

$$\int_{B'_l} |h_n| \, d\mu \le b_n 2^{-2n} \mu(B'_l) \le b_n 2^{-l-n-2} . \tag{8.38}$$

By (8.37) and (8.38), for all $l, n \in \mathbb{N}$, we obtain $\int_{B'_l} |h_n| \, d\mu \le b_n 2^{-l-n-2}$. Thus, for each $l \in \mathbb{N}$,

$$\int_{B'_l} \Big| \sum_{n,k=1}^{\infty} a_{n,k} x_{n,k} \Big| \, d\mu \le \int_{B'_l} \Big| \sum_{n=1}^{\infty} h_n \Big| \, d\mu \le 2^{-l-2} \sum_{n=1}^{\infty} 2^{-n} b_n . \tag{8.39}$$

Given any $r, s \in \mathbb{N}$, let $l = \varphi(r, s)$. We have $B_l \cap A(n,k) = \emptyset$, unless $n = r$ and $k = s$. So, since $\mu(B'_l) \ge \mu(B_l)/2$, we have

$$\int_{B'_l} \Big| \sum_{n,k=1}^{\infty} 2^{-n} \frac{a_{n,k}}{\mu(A(n,k))} \mathbf{1}_{A(n,k)} \Big| \, d\mu \ge \int_{B'_l} \Big| 2^{-r} \frac{a_{r,s}}{\mu(B_l)} \mathbf{1}_{B_l} \Big| \, d\mu \ge 2^{-r-1} |a_{r,s}| .$$

This, together with (8.39), implies that

$$\int_{B'_l} \Big| \sum_{n,k=1}^{\infty} a_{n,k} y_{n,k} \Big| \, d\mu \ge 2^{-r-1} |a_{r,s}| - 2^{-l-2} \sum_{n=1}^{\infty} 2^{-n} b_n .$$

Summing up this inequality over all $r, s \in \mathbb{N}$ and using disjointness of B'_l, we obtain

$$\Big\| \sum_{n,k=1}^{\infty} a_{n,k} y_{n,k} \Big\|_1 \ge \frac{1}{2} \sum_{r,s=1}^{\infty} 2^{-r} |a_{r,s}| - \frac{1}{4} \sum_{n=1}^{\infty} 2^{-n} b_n = \frac{1}{4} \sum_{n,k=1}^{\infty} 2^{-n} |a_{n,k}| .$$

□

Proof of Proposition 8.30

From (8.12) we have that $\|u_n\|_\infty \le 2^{3q(n)+n+2}$. Let $A_n = \{|u_n| \ge 2^{-n+2}\}$. Then (8.15) and (8.14) imply that

$$\mu(A_n) \le 2^{-n-1} \mu(B_n) \le 2^{-3q(n-1)-3n-3} .$$

Define $A'_n = \bigcup_{i>n} A_i$. Then $\mu(A'_n) \le 2^{-3q(n-1)-3n-2}$. Thus, for every $i \le n$,

$$\int_{A'_n} |u_i| \, d\mu 2^{3q(i)+i+2} \mu(A'_n) \le 2^{-2n} . \tag{8.40}$$

Let $L_n = A_n \setminus A'_n$. For each $i > n$, since $|u_i| \le 2^{-i+2}$ on L_i, we have $\int_{L_n} |u_i| \, d\mu \le 2^{-i+2}\mu(L_n) \le 2^{-i+2}$. Thus,

$$\int_{L_n} \Big|\sum_{i \ne n} u_i\Big| \, d\mu \le n2^{-n} + 2^{-n-2} \le 2^{-n}$$

and hence,

$$\int_{L_n} |u_n| \, d\mu \le \int_{L_n} \Big|\sum_{i=1}^{\infty} u_i\Big| \, d\mu + 2^{-n} .$$

Since the sets L_n are disjoint, summing over n gives

$$\sum_{n=1}^{\infty} \int_{L_n} |u_n| \, d\mu \le \Big\|\sum_{i=1}^{\infty} u_i\Big\|_1 + 1 . \tag{8.41}$$

Since $|u_n(t)| \le 2^{-n+2}$, for $t \in [0,1] \setminus A_n$, by (8.40) we have

$$\|u_n\|_1 \le \int_{L_n} |u_n| \, d\mu + 2^{-n+2} + 2^{-2n} .$$

By (8.41), this implies (8.20).

Proof of Proposition 8.31

To prove Proposition 8.31, it is enough to consider the case when each element h_n is a finite sum $\sum_{k=1}^{\infty} a_{n,k} y_{n,k}$. We have to prove that if $\|\sum_{n=1}^{\infty} h_n\|_1 \le 1$ then $\sum_{n=1}^{\infty} \|h_n\|_1 \le 15$. It follows from Proposition 8.29 that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 2^{-n} |a_{n,k}| \le 4$. Since $\|y_{n,k} - x_{n,k}\|_1 = 2^{-n}$, we have $\|\sum_{n=1}^{\infty} u_n\|_1 \le 5$, where $u_n = \sum_{k=1}^{\infty} a_{n,k}x_{n,k}$ for all $n \ge 1$. Since $\sum_{k=1}^{\infty} |a_{n,k}| \le 2^{n+2}$, the element u_n belongs to the closure of $2^{n+2} \operatorname{conv} C_n$. It remains to be seen that

$$\sum_{n=1}^{\infty} \|h_n\|_1 \le \sum_{n=1}^{\infty} \|u_n\|_1 + \sum_{n,k=1}^{\infty} 2^{-n}|a_{n,k}| \le \sum_{n=1}^{\infty} \|u_n\|_1 + 4 ,$$

using Proposition 8.30.

Proof of Proposition 8.32

Let $V : L_1 \to Z$ be an into isomorphism. Without loss of generality we assume that $\|V\| < 1$. For $m \in \mathbb{N}$ and $I \in \mathcal{F}_m$, let $v_I = V(2^m \mathbf{1}_I)$ and choose any $u_I \in U_{v_I}$. Define

$$u_{I,n} = \begin{cases} u_I, & \text{if } n \le m, \\ 2^{m-n} \sum_{\mathcal{F}_m \ni J \subseteq I} u_J, & \text{if } n > m. \end{cases}$$

Obviously, $u_{I,n} \in U_{v_I}$. If I_1 and I_2 are the two distinct elements of $\mathcal{F}_{m+1}$ contained in I, for every $n > m$ we have

$$u_{I,n} = \frac{1}{2}\left(u_{I_1,n} + u_{I_2,n}\right). \tag{8.42}$$

Let $z_I = \varphi((u_{I,n})_{n \ge 1})$. By (8.22), $\|z_I\| \le K$. By (8.21), (8.24) and (8.42), $z_I = \frac{1}{2}(z_{I_1} + z_{I_2})$. By (8.23), $Tz_I = v_I$. Thus, there exists $S \in \mathcal{L}(L_1, Y)$ such that $S(2^m \mathbf{1}_I) = z_I$ for each $m \in \mathbb{N}$ and $I \in \mathcal{F}_m$. Moreover, $\|S\| \le K$ and $V = T \circ S$. Since V is an into isomorphism, T is an isomorphism from $S(L_1)$ onto its image, and thus, T fixes a copy of L_1.

Proof of Proposition 8.33(a)

By our construction, $\|j(x)\| \le 2^{-n}$ for each $n \in \mathbb{N}$ and $x \in D_{q(n)}$. Hence, by Proposition 8.27 and Corollary 8.26, j does not fix a copy of L_1. Suppose that there is an into isomorphism J from L_1 to E. Without loss of generality we assume that $\|J\| < 1$. Let $\delta > 0$ be such that for every $x \in L_1$

$$\|Jx\| \ge \delta \|x\|_1 . \tag{8.43}$$

We will show that j fixes a copy of L_1, which will complete the proof by contradiction.

Let X_n be the Banach space $\bigcup_{\lambda \in \mathbb{R}} \lambda M_n$ endowed with the norm given by the Minkowski functional $\|\cdot\|_n$ of M_n. Let $\widetilde{X} = (\sum_{n=1}^{\infty} X_n)_{\ell_1}$ and $Y = (L_1 \oplus \widetilde{X})_{\ell_\infty}$. Consider the operator $T \in \mathcal{L}(Y, E)$ given by $T(x, (y_n)) = x + \sum_{n=1}^{\infty} y_n$. Observe that T is surjective and $\|T\| \le 1$.

Fix any $z \in E$ with $\|z\|_E < 1$, and let $U_z = T^{-1}(\{z\}) \cap B_Y$.

Consider any sequence $(u_k)_k = (x_k, (y_{n,k})_{n=1}^{\infty})_k \in U_z$. By definition of U_z, for each $k \in \mathbb{N}$, we have

$$z = Tu_k = x_k + \sum_{n=1}^{\infty} y_{n,k} . \tag{8.44}$$

Let $\mathcal{U}$ be any ultrafilter on $\mathbb{N}$. Since M_n is closed convex and bounded in L_1, it is $\sigma(L_1, L_\infty)$-compact, and there exists the $\sigma(L_1, L_\infty)$-limit $y_n = \lim_{k \in \mathcal{U}} y_{n,k}$.

Let $x = z - \sum_{n=1}^{\infty} y_n$. We claim that $\|x\|_1 \le 1$.

Indeed, by Mazur's theorem, for any $m \in \mathbb{N}$, there exists a finitely nonzero sequence $(\alpha_k)_k$ of nonnegative numbers with $\sum_{k=1}^{\infty} \alpha_k = 1$ such that, for all $n \le m$, $\|y_n - \sum_{k=1}^{\infty} \alpha_k y_{n,k}\|_1 \le 2^{-m}$. By (8.44) we get

$$x = z - \sum_{n=1}^{\infty} y_n = \sum_{k=1}^{\infty} \alpha_k \big(x_k + \sum_{n=1}^{\infty} y_{n,k}\big) - \sum_{n=1}^{\infty} y_n ,$$

and thus

$$x - \sum_{k=1}^{\infty} \alpha_k x_k = -\sum_{n=1}^{m} \Big(y_n - \sum_{k=1}^{\infty} \alpha_k y_{n,k}\Big) - \sum_{n=m+1}^{\infty} y_n + \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} \alpha_k y_{n,k} . \quad (8.45)$$

Since $U_z \subset B_Y$, for all $k \in \mathbb{N}$, we have $\|x_k\|_1 \le \|u_k\|_Y \le 1$. Therefore, (8.45) shows that x belongs to the closure in measure of B_{L_1}, and hence $\|x\|_1 \le 1$.

Observe that if we define $\varphi((u_k)_{k=1}^{\infty}) = (x, (y_n)_{n=1}^{\infty}) \in Y$ for each $(u_k)_{k=1}^{\infty} \in U_z$, then (8.21)–(8.24) hold with $K = 1$. Thus, by Proposition 8.32, there exists $S \in \mathcal{L}(L_1, Y)$ such that $J = T \circ S$.

The remaining part of the proof follows from the following lemma.

Lemma 8.39. *For any $m \in \mathbb{N}$, let Y_m be the subspace of Y consisting of all $u = (x, (y_n)_{n=1}^{\infty})$ such that $y_n = 0$ for all $n \ge m$. For every $S \in \mathcal{L}(L_1, Y)$ and every $\varepsilon > 0$, there exist $m \in \mathbb{N}$, $A \in \Sigma^+$ and $S' \in \mathcal{L}(L_1, Y_m)$ such that $\|(S - S')|_{L_1(A)}\| \le \varepsilon$.*

Indeed, if $\varepsilon = \delta/(2\|T\|)$, where $\delta > 0$ is from (8.43), we get that for every $x \in L_1(A)$, $\|T \circ Sx - T \circ S'x\| \le \delta\|x\|/2$. Since $T \circ Sx = Jx$, by (8.43), we get that for every $x \in L_1$, $\|T \circ S'x\| \ge \frac{\delta}{2}\|x\|_1$. Note that $T(Y_m) \subseteq L_1$ and $T : Y_m \to L_1$ is bounded. Considering $T \circ S'$ as valued in L_1, we have for each $x \in L_1(A)$

$$\|j \circ T \circ S'x\| \ge \frac{\delta}{2}\|x\|_1 \ge \frac{\delta}{2\|T \circ S'\|}\|T \circ S'x\|_1 ,$$

and thus, j fixes $T \circ S'(L_1(A))$, which ends the proof of Proposition 8.33(a).

Proof of Lemma 8.39. For any $n \in \mathbb{N}$, let P_n be the canonical projection from Y onto X_n (we identify X_n with the obvious subspace of Y) and $S_n = P_n \circ S$. By the definition of the norm of Y, for each $x \in L_1$ we have

$$\sum_{n=1}^{\infty} \|S_n x\|_n \le \|Sx\|_Y . \quad (8.46)$$

Define $\psi_{n,k} = 2^k \sum_{I \in \mathcal{F}_k} \|S_n \mathbf{1}_I\|_n \mathbf{1}_I$, for each $n, k \in \mathbb{N}$. Then by (8.46), a.e.

$$\sum_{n=1}^{\infty} \psi_{n,k} = 2^k \sum_{I \in \mathcal{F}_k} \sum_{n=1}^{\infty} \|S_n \mathbf{1}_I\|_n \mathbf{1}_I \le \sum_{I \in \mathcal{F}_k} \Big\|S\Big(\frac{\mathbf{1}_I}{\|\mathbf{1}_I\|}\Big)\Big\|_Y \mathbf{1}_I \le \|S\| . \quad (8.47)$$

Recall that a sequence (g_k) in L_1 is called a *submartingale* with respect to (Σ_k) (the latter is an increasing sequence of sub-σ-algebras of Σ) if $\mathbb{E}^{k+1}x_{k+1} \geq x_k$ for each $k \in \mathbb{N}$, see e.g. [18, Definition 10.3.1], and observe that $(\psi_{n,k})_{k=1}^{\infty}$ is a submartingale for each $n \in \mathbb{N}$. Indeed, for each $I \in \mathcal{F}_k$ with $I = I' \sqcup I''$, $I', I'' \in \mathcal{F}_{k+1}$ we have

$$\begin{aligned}\int_I \psi_{n,k}\,\mathrm{d}\mu &= \int_I 2^k \|S_n \mathbf{1}_I\|_n \,\mathrm{d}\mu = \|S_n \mathbf{1}_I\|_n \leq \|S_n \mathbf{1}_{I'}\|_n + \|S_n \mathbf{1}_{I''}\|_n \\ &= \int_{I'} 2^{k+1}\|S_n \mathbf{1}_{I'}\|_n \,\mathrm{d}\mu + \int_{I''} 2^{k+1}\|S_n \mathbf{1}_{I''}\|_n \,\mathrm{d}\mu = \int_I \psi_{n,k+1}\,\mathrm{d}\mu.\end{aligned}$$

By (8.47), $\eta_k \overset{\text{def}}{=} \sum_{n=1}^{\infty} \psi_{n,k} \in L_1$, and (η_k) is a submartingale as well, with $\psi_{n,k} \leq \eta_k$, for each n, k, and $\sup_k \mathbb{E}^k \psi_{n,k} \leq \sup_k \mathbb{E}^k \eta_k \leq \|S\| < \infty$, a.e. for each $n \in \mathbb{N}$. By [18, Theorem 10.3.3], for every $n \in \mathbb{N}$, there exists an a.e. limit $\psi_n = \lim_k \psi_{n,k}$, and moreover, the a.e. limit $\psi = \lim_k \sum_{n=1}^{\infty} \psi_{n,k} = \sum_{n=1}^{\infty} \psi_n$ satisfies $\psi \leq \|S\|$.

For each $n, k \in \mathbb{N}$ and $I \in \mathcal{F}_n$ we have

$$\|S_n \mathbf{1}_I\|_n = \int_I \psi_{n,k}\,\mathrm{d}\mu \leq \int_{[0,1]} \mathbf{1}_I \psi_n \,\mathrm{d}\mu\,.$$

Thus, by the density of simple functions, for every $x \in L_1$, we get $\|S_n x\|_n \leq \int_{[0,1]} |x|\psi_n\,\mathrm{d}\mu$. Hence, for each $m \in \mathbb{N}$ and $x \in L_1$ we have

$$\sum_{n=m}^{\infty} \|S_n x\|_n \leq \int_{[0,1]} |x| \Big(\sum_{n=m}^{\infty} \psi_n\Big)\,\mathrm{d}\mu\,.$$

Since $\sum_{n=1}^{\infty} \psi_n \leq \|S\|$, for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\mu(A) > 0$ where $A = \{\sum_{n=m}^{\infty} \psi_n \leq \varepsilon\}$. Thus, if S' is the composition of S with the canonical projection from Y onto Y_m, for any $x \in L_1(A)$, we have

$$\|Sx - S'x\|_1 \leq \sum_{n=m}^{\infty} \|S_n x\|_n \leq \int_{[0,1]} |x|\,\varepsilon\,\mathrm{d}\mu \leq \varepsilon\|x\|_1\,.$$

□

Proof of Proposition 8.33(b)

Let $u_{n,k} = j(2^n y_{n,k})$. By the definition of $\|\cdot\|_E$, we have $\|u_{n,k}\|_E \leq 2$, since

$$2^n y_{n,k} = \frac{1}{\mu(A(n,k))}\mathbf{1}_{A(n,k)} + 2^n x_{n,k}\,. \tag{8.48}$$

Let $a_{n,k}$ be a finitely nonzero sequence of numbers with $\|\sum_{n,k=1}^{\infty} a_{n,k} u_{n,k}\|_E \leq 1$. By the definition of the norm of E, there are $x \in L_1$ with $\|x\|_1 < 2$, $y_n \in M_n$, and

$\alpha_n \in \mathbb{R}$ with $\sum_{n=1}^{\infty} |\alpha_n| \leq 2$ such that

$$\sum_{n,k=1}^{\infty} a_{n,k} u_{n,k} = x + \sum_{n=1}^{\infty} \alpha_n y_n$$

Without loss of generality we assume that $y_n \in 2^n \operatorname{conv} C_n$, and hence

$$\alpha_n y_n = \sum_{k=1}^{\infty} 2^n b_{n,k} x_{n,k}$$

for some numbers $b_{n,k}$ with $\sum_{k=1}^{\infty} |b_{n,k}| \leq \alpha_n$ for each $n \in \mathbb{N}$. Thus $\sum_{n,k=1}^{\infty} |b_{n,k}| \leq 2$. Using (8.48), we obtain that there exists $x' \in L_1$ with $\|x'\|_1 \leq 4$ such that

$$\sum_{n,k=1}^{\infty} 2^n a_{n,k} y_{n,k} = x' + \sum_{n,k=1}^{\infty} 2^n b_{n,k} y_{n,k} \ ,$$

or equivalently, $\sum_{n,k=1}^{\infty} 2^n \big(a_{n,k} - b_{n,k}\big) y_{n,k} = -x'$.

Since $\|x'\|_1 \leq 4$, Proposition 8.29 implies that $\sum_{n,k=1}^{\infty} |a_{n,k} - b_{n,k}| < 16$ and so, $\sum_{n,k=1}^{\infty} |a_{n,k}| < 18$. Thus, the closed linear span of $(u_{n,k})_{n,k\geq 1}$, which is the closure of $j(X)$, is isomorphic to ℓ_1.

Proof of Proposition 8.33(c)

Let $\tau^X : L_1 \to L_1/X$ and $\tau^W : E \to E/W$ be the quotient maps. Since $\tau^W \circ j|_X = 0$, there is a continuous linear operator $T : L_1/X \to E/W$ such that $T \circ \tau^X = \tau^W \circ j$. Since $j(L_1)$ is dense in E, the image $T(L_1/X)$ is dense in E/W and hence, it is left to show that T is an into isomorphism.

Consider any $u \in L_1$ with $\|T \circ \tau^X u\| < 1$. Then $\|\tau^W \circ ju\| < 1$, and so, there exists $v \in X$ such that $\|j(u) - j(v)\|_E < 1$. Thus

$$u - v = x + \sum_{n=1}^{\infty} \alpha_n y_n,$$

where $x \in B_{L_1}$, $\alpha_n \in \mathbb{R}$ with $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$ and $y_n \in M_n$. By Proposition 8.31, $\sum_{n=1}^{\infty} |\alpha_n| \|y_n\| < \infty$, and hence we can write

$$u - v = x' + \sum_{n=1}^{N} \alpha_n y_n,$$

where $\|x'\| \leq 2$. Then we have that $y_n \in 2^n \operatorname{conv} C_n$ for each $n = 1, \dots, N$. By (8.48), $2^n x_{n,k} \in B_{L_1} + X$, and thus, $2^n \operatorname{conv} C_n \subseteq B_{L_1} + X$. Since $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$, we have that $\sum_{n=1}^{N} \alpha_n y_n \in B_{L_1} + X$. Thus, there exists $v' \in X$ such that $\|u - v - v'\|_1 \leq 3$. This shows that $\|\tau^X u\| \leq 3$, finishing the proof that T is an into isomorphism.

Chapter 9

Spaces X for which every operator $T \in \mathcal{L}(L_p, X)$ is narrow

The classical Pitt theorem [109] (see also [79, p. 76]) asserts that, for any $1 \le p < r < \infty$, every operator $T \in \mathcal{L}(\ell_r, \ell_p)$ is compact, and every operator $T \in \mathcal{L}(\ell_r, c_0)$ is compact. In this chapter we are interested in analogs of Pitt's theorem for narrow operators. That is, we want to identify classes of spaces X, so that every operator $T \in \mathcal{L}(L_p, X)$ is narrow. In Section 9.1 we recall some characterizations of spaces so that every operator $T \in \mathcal{L}(L_1, X)$ is narrow, which are presented elsewhere in this book, and we show a characterization due to V. Kadets and Popov [56] in terms of ranges of vector measures. In Section 9.2 we show that every $T \in \mathcal{L}(E, c_0(\Gamma))$ is narrow, where E belongs to a large class of spaces including L_p for all p, $0 < p < \infty$. Section 9.3 gives the closest analog of Pitt's theorem, namely it gives the characterization of parameters p, r so that every operator $T \in \mathcal{L}(L_p, L_r)$ is narrow. Section 9.4 addresses the same question for operators $T \in \mathcal{L}(L_p, \ell_r)$. This is the most involved section in this chapter. It combines a geometric argument and a probabilistic method which uses the martingale structure of the partial sums of the Haar system, stopping times and the central limit theorem (although the notions of martingales and stopping times are not explicitly mentioned). These ideas were first used in this context by V. Kadets and Schechtman in [59], and later also by V. Kadets, Kalton and Werner in [53] (cf. Section 11.1). In the last section we use methods developed in Section 9.4 to study ℓ_2-strictly singular operators on L_p. We partially answer a question of Plichko and Popov [110] (cf. Chapter 7) by proving that every ℓ_2-strictly singular operator from L_p to a space X with an unconditional basis, is narrow. Another, incomparable, partial answer to this problem is presented in Section 10.9. Results of Sections 9.4 and 9.5 come from [102].

9.1 A characterization using the ranges of vector measures

In this section we study spaces X so that every operator from L_1 to X is narrow.

We already saw a characterization of such spaces: Rosenthal's Theorem 8.4 says that these spaces are exactly the spaces X so that L_1 does not sign-embed in X. Later V. Kadets, Kalton and Werner [53] proved that if X has an unconditional basis then every operator $T \in \mathcal{L}(L_1, X)$ is narrow, and that this implies that L_1 cannot be sign-embedded in a Banach space with an unconditional basis, which was announced earlier by Rosenthal without published proof, see Section 11.1 (Theorem 11.11).

The class of spaces with the property that every operator from L_1 to X is narrow includes spaces X which have the RNP. Indeed, we saw that every representable operator is narrow (Proposition 2.4) and it is well known that if X has the RNP then every operator $T \in \mathcal{L}(L_1, X)$ is representable [29, p. 63]. Also it follows from Theorem 8.22 that every operator from L_1 to a separable dual space is narrow, and even the larger class $\mathfrak{G}$ is identified so that every operator $T \in \mathcal{L}(L_1, X)$ is narrow for every $X \in \mathfrak{G}$ (see the discussion before Theorem 8.22).

The goal of this section is to give a characterization of Banach spaces X for which every operator $T \in \mathcal{L}(L_1, X)$ is narrow in terms of ranges of vector measures.

First we recall some definitions. Let (Ω, Σ) be a *measurable space*, that is, a set Ω and a σ-algebra Σ of subsets of Ω. Let X be a Banach space. A map $\mathbf{m} : \Sigma \to X$ is called a *σ-additive* (= countably additive) *vector measure* if $\mathbf{m}(\bigsqcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbf{m}(A_n)$ for each sequence of disjoint sets $A_n \in \Sigma$. Given a σ-additive vector measure $\mathbf{m} : \Sigma \to X$, the *variation* of $\mathbf{m}$ is the positive scalar σ-additive measure $|\mathbf{m}| : \Sigma \to [0, \infty]$ defined by

$$|\mathbf{m}|(A) = \sup\Big\{\sum_{k=1}^{n} \|\mathbf{m}(A_k)\| : n \in \mathbb{N},\ A_k \in \Sigma,\ A = \bigsqcup_{k=1}^{n} A_k\Big\}$$

for each $A \in \Sigma$. A vector measure $\mathbf{m} : \Sigma \to X$ is said to have *bounded variation* if $\mathbf{m}(\Omega) < \infty$ (in this case $\mathbf{m}(A) < \infty$ for each $A \in \Sigma$). Otherwise, the measure is said to have *unbounded variation*. For example, $\mathbf{m}(A) = \mathbf{1}_A$, $A \in \Sigma$ is a σ-additive vector measure taking values in $L_p(\mu)$ where (Ω, Σ, μ) is a measure space and $1 \le p < \infty$, and is not σ-additive if we consider its values in $L_\infty(\mu)$. Notice that this measure has bounded variation if $p = 1$, and unbounded variation if $1 < p < \infty$. For finite dimensional X, any X-valued σ-additive measure has bounded variation. We refer to Diestel and Uhl's book [29] (1977) for more information on vector measures.

The classical Lyapunov theorem [86] asserts that if X is finite dimensional then the range of any X-valued measure σ-additive vector measure is convex. The converse is also true [29, p. 265]: if the range of any X-valued measure σ-additive vector measure of bounded variation is convex then X is finite dimensional. Here is a simple proof: let $\dim X = \infty$ and $T \in \mathcal{L}(L_1, X)$ be any injective operator, then $\mathbf{m}(A) = T\mathbf{1}_A$ is an X-valued σ-additive vector measure of bounded variation with nonconvex range. Indeed, putting $x = \mathbf{m}([0,1])$, we obtain that $x/2$ does not belong to the range of $\mathbf{m}$, because otherwise $\mathbf{m}(A) = x/2$ would imply that $\mathbf{m}([0,1] \setminus A) = \mathbf{m}([0,1]) - \mathbf{m}(A)$, and hence, $T(\mathbf{1}_A - \mathbf{1}_{([0,1]\setminus A)}) = x/2 - x/2 = 0$. This contradicts the injectivity of T.

It is natural to ask for which infinite dimensional spaces X, every X-valued σ-additive vector measure of bounded variation has a convex range closure. The above example shows that this will not hold if there exists an injective operator from L_1 to X. It turns out that the characterization of such spaces can be formulated in terms of narrow operators. The following result was obtained simultaneously and independently by V. Kadets and Popov.

Theorem 9.1 ([56]). *A Banach space X has the property that every X-valued σ-additive vector measure of bounded variation has convex range closure if and only if* *every operator* $T \in \mathcal{L}(L_1, X)$ *is narrow (which, by Theorem* 8.4 *is equivalent to the condition that* L_1 *does not sign-embed in* X*).*

For the proof we will need the following lemma.

Lemma 9.2. *Let* $\mathbf{m}$ *be an* X*-valued* σ*-additive vector measure of bounded variation. If for every* $A \in \Sigma$ *and every* $\varepsilon > 0$ *there exists* $B \in \Sigma(A)$ *such that*

$$\left\|\mathbf{m}(B) - \frac{1}{2}\mathbf{m}(A)\right\| < \varepsilon ,$$

then the range $\mathbf{m}(\Sigma)$ *of the measure* $\mathbf{m}$ *has convex closure.*

Proof of Lemma 9.2. Using induction on n, we prove the following statement: for any $n \in \mathbb{N}$, $k = 1, \ldots, 2^n$, $A \in \Sigma$ and $\varepsilon > 0$ there exists $B \in \Sigma(A)$ such that

$$\left\|\mathbf{m}(B) - \frac{k}{2^n}\mathbf{m}(A)\right\| < \varepsilon .$$

The induction base is given by the assumptions of the lemma. Suppose that the statement is true for a given $n \geq 1$. Fix $k \leq 2^{n+1}$, $A \in \Sigma$ and $\varepsilon > 0$. Let $B_1 \in \Sigma(A)$ be such that

$$\left\|\mathbf{m}(B_1) - \frac{1}{2}\mathbf{m}(A)\right\| < \frac{2^{n-1}\varepsilon}{k} .$$

We consider the following two cases:

(1) Assume that $1 \leq k \leq 2^n$. We choose $B \in \Sigma(B_1)$ so that

$$\left\|\mathbf{m}(B) - \frac{k}{2^n}\mathbf{m}(B_1)\right\| < \frac{\varepsilon}{2} .$$

Then,

$$\begin{aligned}\left\|\mathbf{m}(B) - \frac{k}{2^{n+1}}\mathbf{m}(A)\right\| &\leq \left\|\mathbf{m}(B) - \frac{k}{2^n}\mathbf{m}(B_1)\right\| + \left\|\frac{k}{2^n}\mathbf{m}(B_1) - \frac{k}{2^{n+1}}\mathbf{m}(A)\right\| \\ &< \frac{\varepsilon}{2} + \frac{k}{2^n} \cdot \frac{2^{n-1}\varepsilon}{k} = \varepsilon.\end{aligned}$$

(2) Assume that $2^n < k \leq 2^{n+1}$. We choose $B_2 \in \Sigma(B_1)$ so that

$$\left\|\mathbf{m}(B_2) - \frac{k-2^n}{2^n}\mathbf{m}(B_1)\right\| < \frac{\varepsilon}{2} ,$$

and let $B = (A \setminus B_1) \cup B_2$. Then,

$$\begin{aligned}\left\|\mathbf{m}(B) - \frac{k}{2^{n+1}}\mathbf{m}(A)\right\| &= \left\|\mathbf{m}(A) - \mathbf{m}(B_1) + \mathbf{m}(B_2) - \frac{1}{2}\mathbf{m}(A) - \frac{k-2^n}{2^{n+1}}\mathbf{m}(A)\right\| \\ &\leq \left\|\frac{1}{2}\mathbf{m}(A) - \mathbf{m}(B_1)\right\| + \left\|\mathbf{m}(B_2) - \frac{k-2^n}{2^n}\mathbf{m}(B_1)\right\| \\ &\quad + \frac{k-2^n}{2^n}\left\|\mathbf{m}(B_1) - \frac{1}{2}\mathbf{m}(A)\right\| \\ &< \frac{2^{n-1}\varepsilon}{k}\Big(1 + \frac{k-2^n}{2^n}\Big) + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, the statement is proved by induction. Since the set of all dyadic numbers $k/2^n$ is dense in $[0, 1]$, we obtain that for any $t \in [0, 1]$, $A \in \Sigma$ and $\varepsilon > 0$ there exists $B \in \Sigma(A)$ such that

$$\|\mathbf{m}(B) - t\mathbf{m}(A)\| < \varepsilon \, . \tag{9.1}$$

Fix any $A, B \in \Sigma$, $t \in [0, 1]$ and $\varepsilon > 0$. Using (9.1), we choose $A_1 \subseteq A \setminus B$ so that $\|\mathbf{m}(A_1) - t\mathbf{m}(A \setminus B)\| < \varepsilon/2$, and $B_1 \subseteq B \setminus A$ so that $\|\mathbf{m}(B_1) - (1-t)\mathbf{m}(B \setminus A)\| < \varepsilon/2$. Then for $C = A_1 \cup B_1 \cup (A \cap B)$ we get

$$\begin{aligned}
&\|\mathbf{m}(C) - t\mathbf{m}(A) - (1-t)\mathbf{m}(B)\| \\
&= \|\mathbf{m}(A_1) + \mathbf{m}(B_1) + \mathbf{m}(A \cap B) - t\mathbf{m}(A \setminus B) - (1-t)\mathbf{m}(B \setminus A) - \mathbf{m}(A \cap B)\| \\
&\le \|\mathbf{m}(A_1) - t\mathbf{m}(A \setminus B)\| + \|\mathbf{m}(B_1) - (1-t)\mathbf{m}(B \setminus A)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square
\end{aligned}$$

Proof of Theorem 9.1. $\Rightarrow$. Suppose $T \in \mathcal{L}(L_1, X)$. Then $\mathbf{m}(A) = T\mathbf{1}_A$ is an X-valued σ-additive measure of bounded variation. Fix an arbitrary $A \in \Sigma$ and consider the restriction of $\mathbf{m}$ to $\Sigma(A)$. Since every X-valued σ-additive vector measure of bounded variation has convex range closure, the element

$$\frac{1}{2} T\mathbf{1}_A = \frac{1}{2}\big(\mathbf{m}(A) + \mathbf{m}(\emptyset)\big)$$

can be approximated by values of $\mathbf{m}$, so for each $\varepsilon > 0$ there exists $A_\varepsilon \in \Sigma(A)$ with

$$\Big\| \frac{1}{2} T\mathbf{1}_A - T\mathbf{1}_{A_\varepsilon} \Big\| < \frac{\varepsilon}{2} \, .$$

We have

$$\big\| T(\mathbf{1}_{A \setminus A_\varepsilon} - \mathbf{1}_{A_\varepsilon}) \big\| = \big\| T(\mathbf{1}_A - 2\mathbf{1}_{A_\varepsilon}) \big\| < \varepsilon \, .$$

Thus T is narrow.

$\Leftarrow$. Assume $\mathbf{m}$ is a σ-additive X-valued measure of bounded variation. We define an operator $T : L_1([0, 1], \Sigma, |\mathbf{m}|) \to X$ by setting $T\mathbf{1}_A = \mathbf{m}(A)$ for any $A \in \Sigma$. Then for any simple function $x = \sum_{k=1}^m a_k \mathbf{1}_{A_k}$, where $[0, 1] = \bigsqcup_{k=1}^m A_k$ we have that

$$\|Tx\| = \Big\| \sum_{k=1}^m a_k \mathbf{m}(A_k) \Big\| \le \sum_{k=1}^m |a_k| \cdot |\mathbf{m}|(A_k) = \|x\| \, .$$

Thus the operator T can be extended by linearity and continuity to the whole space $L_1([0, 1], \Sigma, |\mathbf{m}|)$.

Since every operator from L_1 to X is narrow, for an arbitrary finite atomless measure space (Ω, Σ, μ), every operator from $L_1(\mu)$ to X is also narrow. Indeed, if $A \in \Sigma^+$, then we can construct a sub-σ-algebra $\Sigma_1 \subseteq \Sigma(A)$ and a measure-preserving map (see Carathéodory's theorem 1.16), up to the multiple of $\mu(A)$, from $[0, 1]$ onto A, such that the range of Σ is Σ_1. Hence, the restriction of every operator

$S \in \mathcal{L}(L_1(\mu), X)$ to the subspace $L_1(A, \Sigma_1, \mu|_{\Sigma_1})$ is a narrow operator. Thus, there exists $x \in L_1(\mu)$ such that $x^2 = \mathbf{1}_A$, $\int_\Omega x\, d\mu = 0$ and $\|Sx\| < \varepsilon$.

Thus, T is a narrow operator. Choose $y \in L_1([0,1], \Sigma, |\mathbf{m}|)$ so that $y^2 = \mathbf{1}_A$, $\int_\Omega y d|\mathbf{m}| = 0$ and $\|Ty\| < \varepsilon$. Then for $B = \{t \in A : y(t) = 1\}$ we obtain

$$\begin{aligned}\Big\|\mathbf{m}(B) - \frac{1}{2}\mathbf{m}(A)\Big\| &= \Big\|\mathbf{m}(B) - \frac{1}{2}\mathbf{m}(B) - \frac{1}{2}\mathbf{m}(A \setminus B)\Big\| \\ &= \frac{1}{2}\big\|\mathbf{m}(B) - \mathbf{m}(A \setminus B)\big\| = \frac{1}{2}\|Tx\| < \varepsilon.\end{aligned}$$

By Lemma 9.2, $\mathbf{m}$ has convex range closure. □

9.2 Every operator from E to $c_0(\Gamma)$ is narrow

The following theorem is valid for the case of the real scalar field.

Theorem 9.3. *Let E be a Köthe F-space over the reals on a finite atomless measure space (Ω, Σ, μ) for which there exists a reflexive Köthe–Banach space E_1 on (Ω, Σ, μ) with continuous inclusion embedding $E_1 \subseteq E$. Let Γ be an arbitrary set. Then every operator $T \in \mathcal{L}(E, c_0(\Gamma))$ is narrow.*

For the proof, we need the following simple observation concerning linear operators on vector spaces.

Lemma 9.4. *Let $S : X \to Y$ be a linear operator acting between linear spaces. If a linear subspace $Y_0 \subseteq Y$ has finite codimension in Y then $X_0 = T^{-1}Y_0$ has finite codimension in X.*

Proof of Lemma 9.4. Let m be the codimension of Y_0 in Y, and $x_1, \dots, x_{m+1}$ be any vectors in X. Since $\dim Y/Y_0 = m$, there are scalars $(a_i)_{i=1}^{m+1}$ with $\sum_{k=1}^{m+1} |a_k| > 0$ such that $\sum_{k=1}^{m+1} a_k T x_k \in Y_0$. But since $\sum_{k=1}^{m+1} a_k x_k \in Y_0$, we conclude that $\dim X/X_0 \le m$ by arbitrariness of $x_1, \dots, x_{m+1} \in X$. □

Proof of Theorem 9.3. Fix any $\varepsilon > 0$ and set

$$K = \Big\{x \in B_{L_\infty(\mu)} : \int_\Omega x\, d\mu = 0 \ \& \ \|Tx\| \le \varepsilon\Big\}.$$

Observe that K is a nonempty convex and closed subset of E. Moreover, K is bounded in E_1, and by continuity of the embedding $E_1 \subseteq E$ we obtain that K is closed in E_1. Thus, by the Banach–Alaoglu theorem, K is a nonempty convex weakly compact subset of E_1. By the Krein–Milman theorem, there exists an extreme point $x_0 \in K$. We prove that $|x_0(t)| = 1$ for almost all $t \in \Omega$. Suppose on the contrary, that there exists $\delta > 0$ and $B \in \Sigma^+$ such that $|x_0(t)| \le 1 - \delta$ for all $t \in B$. Let $Tx_0 = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma$ where $(a_\gamma)_{\gamma \in \Gamma}$ are scalars with $\lim_{\gamma \in \Gamma} a_\gamma = 0$, and $(e_\gamma)_{\gamma \in \Gamma}$ is the unit vector basis of $c_0(\Gamma)$ with the biorthogonal functionals $(e_\gamma^*)_{\gamma\in\Gamma}$. Choose a finite set $\Gamma_0 \subset \Gamma$ so that $|a_\gamma| < \varepsilon/2$ for each $\gamma \in \Gamma \setminus \Gamma_0$. By Lemma 9.4, the subspace $T^{-1}([e_\gamma]_{\gamma\in\Gamma\setminus\Gamma_0})$ has finite codimension in E. Since the intersection of two finite codimensional subspaces is a finite codimensional subspace as well, the subspace

$$X = T^{-1}\Big([e_\gamma]_{\gamma\in\Gamma\setminus\Gamma_0}\Big) \cap \Big\{x \in E : \int_\Omega x \,\mathrm{d}\mu = 0\Big\}$$

has finite codimension in E. By Lemma 2.9, $L_\infty(B) \cap X \neq \{0\}$. We choose any $x \in L_\infty(B) \cap X$, $x \neq 0$, such that $\|x\|_\infty \le \delta$ and $\|Tx\| \le \varepsilon/2$. Now it is easy to verify that both $(x_0 + x) \in K$ and $(x_0 - x) \in K$. Indeed, since $x \in X$ and $x_0 \in K$, we have that $\int_\Omega (x_0 \pm x)\,\mathrm{d}\lambda = 0$. Since $|x_0(t)| < 1 - \delta$ for all $t \in A$, $x \in L_\infty(A)$ and $\|x\|_\infty \le \delta$, we obtain that $(x_0 \pm x) \in B_{L_\infty(\mu)}$. Observe that then

$$\big|e_\gamma^*(Tx_0 \pm x)\big| \begin{cases} = |a_\gamma| \le \varepsilon, & \text{if } \gamma \in \Gamma_0 \\ \le |a_\gamma| + |e_\gamma^*(Tx)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, & \text{if } \gamma \in \Gamma \setminus \Gamma_0. \end{cases}$$

Thus, $\|T(x_0 \pm x)\| \le \varepsilon$ and hence, $(x_0 \pm x) \in K$. This contradicts the fact that x_0 is an extreme point of K. □

Remark 9.5. Observe that we proved that both in the real and the complex case for every $\varepsilon > 0$ there exists $x \in E$ such that $|x| = \mathbf{1}_\Omega$, $\int_\Omega x\,\mathrm{d}\mu = 0$ and $\|Tx\| < \varepsilon$. In general, such an x is called a complex sign.

By Theorem 3.5 we obtain the following consequence.

Corollary 9.6. *Let E be a strictly nonconvex Köthe F-space over the reals on a finite atomless measure space (Ω, Σ, μ) for which there exists a reflexive Köthe–Banach space E_1 on (Ω, Σ, μ) with continuous inclusion embedding $E_1 \subseteq E$. Let Γ be an arbitrary set. Then $\mathcal{L}(E, c_0(\Gamma)) = \{0\}$.*

Note that for $E = L_p$ with $0 < p < 1$, Kalton proved much more, that for any topological vector space X every ℓ_2-strictly singular operator $T \in \mathcal{L}(L_p, X)$ is zero [65]. Hence, if X contains no isomorph of ℓ_2 then $\mathcal{L}(L_p, X) = \{0\}$.

9.3 An analog of the Pitt compactness theorem for L_p-spaces

The following result of V. Kadets and Popov [57], is an analog of Pitt's compactness theorem for narrow operators on the Lebesgue spaces L_p.

Theorem 9.7. *If $1 \le p < 2$ and $p < r < \infty$ then every operator $T \in \mathcal{L}(L_p, L_r)$ is narrow.*

We remark that Theorem 9.7 is false for any other values of p and r. If $p \geq 2$ then the composition $J_r \circ I_{p,2}$ of the identity embedding $I_{p,2} : L_p \to L_2$ and the isomorphic embedding $J_r : L_2 \to L_r$ is evidently not narrow. And if $1 \leq p < 2$ and $1 \leq r \leq p$ then the identity embedding of L_p into L_r is not narrow.

Before the proof, we recall that a Banach space X is said to have *infratype* $q > 1$ if there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$,

$$\min_{\theta_k = \pm 1} \Big\| \sum_{k=1}^{n} \theta_k x_k \Big\| \leq C \Big(\sum_{k=1}^{n} \|x_k\|^q \Big)^{1/q} .$$

Maurey and Pisier [95] showed that for every $q > 1$, L_q has infratype $\min\{q, 2\}$ (for $1 \leq q \leq 2$, cf. also Lemma 7.63(b)). Thus, Theorem 9.7 follows from the next more general result.

Theorem 9.8. *Let $1 \leq p < 2$ and X be a Banach space with infratype $q > p$. Then every operator from L_p to X is narrow.*

Proof. Assume $T \in \mathcal{L}(L_p, X)$, $A \in \Sigma$, $n \in \mathbb{N}$. We decompose A into n subsets of equal measure $A_1, \dots, A_n$.

Then

$$\begin{aligned} \min_{\theta_k = \pm 1} \Big\| \sum_{k=1}^{n} \theta_k T \mathbf{1}_{A_k} \Big\| &\leq C \Big(\sum_{k=1}^{n} \|T \mathbf{1}_{A_k}\|^q \Big)^{1/q} \\ &\leq C \|T\| \Big(\sum_{k=1}^{n} \|\mathbf{1}_{A_k}\|^q \Big)^{1/q} \leq C \|T\| \Big(\sum_{k=1}^{n} \Big(\frac{\lambda(A)}{n} \Big)^{q/p} \Big)^{1/q} \\ &\leq C \|T\| \big(\lambda(A)\big)^{1/p} \Big(n^{1-q/p} \Big)^{1/q} \to 0 \text{ as } n \to \infty . \end{aligned}$$

This implies that for every $\varepsilon > 0$ there exist n and sign numbers $\theta_1, \dots, \theta_n$ so that for

$$x = \sum_{k=1}^{n} \theta_k \mathbf{1}_{A_k}$$

we have $\|Tx\| < \varepsilon$. By Proposition 1.9, this implies that T is narrow. □

9.4 When is every operator from L_p to ℓ_r narrow?

The main theorem of this section is Theorem 9.9 which characterizes all values of the parameters $1 \leq p, q < \infty$ for which every operator $T \in \mathcal{L}(L_p, \ell_r)$ is narrow. We finish the section with a few corollaries that provide additional examples of spaces X so that every operator from L_p, for $p > 2$, to X is narrow. We also prove that despite the existence of a nonnarrow operator from L_p to ℓ_2 if $p > 2$, there is no sign-embedding from L_p, $p > 2$, to ℓ_2.

Let $2 \le p < \infty$, $I_{p,2} : L_p \to L_2$ be the identity embedding, and $S : L_2 \to \ell_2$ be an isomorphism. Then, obviously, the composition $S \circ I_{p,2} : L_p \to L_2$ is not narrow. The following theorem asserts that for all other values of the parameters p, r every operator $T \in \mathcal{L}(L_p, \ell_r)$ is narrow.

Theorem 9.9. *Let $1 \le p, r < \infty$, and assume that either $r \ne 2$ or $r = 2$ and $p < 2$. Then every operator $T \in \mathcal{L}(L_p, \ell_r)$ is narrow.*

In our proof we consider separately the following cases:

(i) $1 \le p < 2$ and $p < r$;

(ii) $1 \le r < 2$;

(iii) $p \ge 2$ and $r > 2$.

Case (i) follows easily from Theorem 9.7; case (ii) is a simple observation from [110], and case (iii) is a deep result from [102].

Proof of Theorem 9.9 *in case* (ii). Let $1 \le r < 2$, $T \in \mathcal{L}(L_p, \ell_r)$, $A \in \Sigma^+$ and $\varepsilon > 0$. Consider a Rademacher system (r_n) in $L_p(A)$. By Khintchine's inequality, $[r_n]$ is isomorphic to ℓ_2, and by Pitt's theorem, the restriction $T|_{[r_n]}$ is compact. Since (r_n) tends weakly to 0, we have that $\lim_{n\to\infty} Tr_n = 0$. Thus there exists $n \in \mathbb{N}$ with $\|Tr_n\| < \varepsilon$. It remains to observe that $r_n^2 = \mathbf{1}_A$ and $\int_{[0,1]} r_n \, d\mu = 0$. □

The proof of Theorem 9.9 in case (iii) is much more involved. The idea is to first prove that a generic operator $T \in \mathcal{L}(L_p, \ell_r)$ can be replaced with the operator $S \in \mathcal{L}(L_p, \ell_r)$ which sends the normalized Haar system in L_p to the unit vector basis of ℓ_r (Proposition 9.10). We then prove that, when $r \ge p$, the operator S is narrow by constructing a sign which S sends to a vector of small norm. The construction of this sign is probabilistic in nature and uses the martingale structure of the partial sums of the Haar system, stopping times and the central limit theorem (although the notions of martingales and stopping times are not explicitly mentioned). Similar ideas were first used in this context by V. Kadets and Schechtman in [59], and later also by V. Kadets, Kalton and Werner in [53] (cf. Section 11.1). Our proof follows [102].

For the rest of the section we use the following notation:

- $(\overline{h}_n)_{n=1}^\infty$ – the L_∞-normalized Haar system;
- (h_n) and (h_n^*) – the L_p- and L_q-normalized Haar functions, respectively, where $1/p + 1/q = 1$.

Proposition 9.10. *Suppose $1 \le p < \infty$, X is a Banach space with an unconditional basis (x_n), $T \in \mathcal{L}(L_p, X)$ satisfies $\|Tx\| \ge 2\delta$ for each mean zero sign $x \in L_p$ on $[0,1]$ and some $\delta > 0$. Then there exists an operator $S \in \mathcal{L}(L_p, X)$, a normalized* *block basis (u_n) of (x_n) and real numbers (a_n) such that*

(a) $Sh_n = a_n u_n$ *for each* $n \in \mathbb{N}$ *with* $a_1 = 0$;

(b) $\|Sx\| \geq \delta$, *for each mean zero sign* $x \in L_p$ *on* $[0, 1]$;

(c) *There exists a linear isometry* V *of* L_p *into* L_p, *which signs sends to signs, so that* $\|Sx\| \leq \|TVx\| + 2\delta$ *for every* $x \in L_p$ *with* $\|x\| = 1$.

If, moreover, $\|Tx\| \geq 2\delta\|x\|$ *for every sign* x, *then* $|a_n| \geq \delta$ *for each* $n \geq 2$.

Proof. Let $(P_n)_{n=1}^{\infty}$ be the basis projections in X with respect to the basis (x_n) and $P_0 = 0$. First we construct an operator $\widetilde{S}$ which has all the desired properties of S, with the small difference that $\widetilde{S}$ is defined on the closed linear span of a sequence which is isometrically equivalent to the Haar system. For this purpose, we construct a sequence of integers $0 = s_1 < s_2 < \ldots$, a tree $(A_{m,k})_{m=0}^{\infty}{}_{k=1}^{2^m}$ of measurable sets $A_{m,k} \subseteq [0, 1]$ and an operator $\widetilde{S} \in \mathcal{L}(L_p(\Sigma_1), X)$, where Σ_1 is the sub-σ-algebra of Σ generated by the $A_{m,k}$ with the following properties:

(P1) $A_{0,1} = [0, 1]$;

(P2) $A_{m,k} = A_{m+1,2k-1} \sqcup A_{m+1,2k}$, $\mu(A_{m,k}) = 2^{-m}$ for all m, k;

(P3) $\|\widetilde{S}x\| \geq \delta$ for each mean zero sign $x \in L_p(\Sigma_1)$ on $[0, 1]$;

(P4) if $h'_1 = \mathbf{1}$ and $h'_{2^m+k} = 2^{m/p}\big(\mathbf{1}_{A_{m+1,2k-1}} - \mathbf{1}_{A_{m+1,2k}}\big)$ for all m, k, then we have $\widetilde{S}h'_1 = 0$ and $\widetilde{S}h'_n = (P_{s_n} - P_{s_{n-1}})Th'_n$ for $n = 2, 3, \ldots$.

We will use the following convention: once the sets $A_{m+1,2k-1}$ and $A_{m+1,2k}$ are defined for given m, k, we consider h'_{2^m+k} to be defined by the equality from (P4). To construct a family with the above properties, we set $A_{1,1} = [0, 1/2)$ and $A_{1,2} = [1/2, 1]$. Then choose $s_2 > 1$ so that

$$(C_2) \qquad \big\|Th'_2 - \big(P_{s_2} - P_{s_1}\big)Th'_2\big\| = \big\|Th'_2 - P_{s_2}Th'_2\big\| \leq \frac{\delta}{2}.$$

Since the operator $P_{s_2}T$ is finite rank and hence narrow, there exists a mean zero sign $x_{1,1}$ on the set $A_{1,1}$ such that $\|P_{s_2}Tx_{1,1}\| \leq \frac{\delta}{8 \cdot 2^{1/p}}$. Then set $A_{2,1} = x_{1,1}^{-1}(1) = \{t \in [0, 1] : x_{1,1}(t) = 1\}$, $A_{2,2} = x_{1,1}^{-1}(1)$ and observe that $h'_3 = 2^{1/p}x_{1,1}$ and $\|P_{s_2}Th'_3\| \leq \delta/8$. Now we choose $s_3 > s_2$ so that $\|Th'_3 - P_{s_3}Th'_3\| \leq \delta/8$. Then we have

$$(C_3) \qquad \big\|Th'_3 - \big(P_{s_3} - P_{s_2}\big)Th'_3\big\| \leq \frac{\delta}{4}.$$

Since $P_{s_3}T$ is narrow, there is a mean zero sign $x_{1,2}$ on $A_{1,2}$ such that $\|P_{s_3}Tx_{1,2}\| \leq \frac{\delta}{16 \cdot 2^{1/p}}$. Put $A_{2,3} = x_{1,2}^{-1}(1)$ and $A_{2,4} = x_{1,2}^{-1}(-1)$. Observe that $h'_4 = 2^{1/p}x_{1,2}$ and $\|P_{s_3}Th'_4\| \leq \delta/16$. Choose $s_4 > s_3$ so that $\|Th'_4 - P_{s_4}Th'_4\| \leq \delta/16$. Then

$$(C_4) \qquad \big\|Th'_4 - \big(P_{s_4} - P_{s_3}\big)Th'_4\big\| \leq \frac{\delta}{8}.$$

Further we analogously find a mean zero sign $x_{2,1}$ on $A_{2,1}$ such that $\|P_{s_4}Tx_{2,1}\| \le \frac{\delta}{32\cdot 2^{2/p}}$. Then putting $A_{3,1} = x_{2,1}^{-1}(1)$, $A_{3,2} = x_{2,1}(-1)$, we obtain $h'_5 = 2^{2/p}x_{2,1}$ and $\|P_{s_4}Th'_5\| \le \delta/32$. Now choose $s_5 > s_4$ so that $\|Th'_5 - P_{s_5}Th'_5\| \le \delta/32$, and obtain

$$(C_5) \qquad \|Th'_5 - \big(P_{s_5} - P_{s_4}\big)Th'_5\| \le \frac{\delta}{16}.$$

Continuing the procedure, we construct a sequence of integers $0 = s_1 < s_2 < \ldots$, a tree $(A_{m,k})_{m=0}^{\infty}{}_{k=1}^{2^m}$ of measurable sets $A_{n,k} \subseteq [0,1]$ which satisfies conditions (P1) and (P2), for which we have

$$(C_n) \qquad \Big\|Th'_n - \big(P_{s_n} - P_{s_{n-1}}\big)Th'_n\Big\| \le \frac{\delta}{2^{n-1}}.$$

Note that property (P4) defines the operator $\widetilde{S}$ on the system (h'_n). We show that $\widetilde{S}$ could be extended by linearity and continuity on $L_p(\Sigma_1)$. Let $x = \sum_{n=1}^{N} \beta_n h'_n \in L_p(\Sigma_1)$ with $\|x\| = 1$. Note that $|\beta_n| \le 2$, because the Haar system is a monotone basis in L_p. Then by (C_n), we obtain

$$\begin{aligned}\|\widetilde{S}x\| &= \Big\|\sum_{n=2}^{N} \beta_n (P_{s_n} - P_{s_{n-1}})Th'_n\Big\| \\ &\le \Big\|\sum_{n=2}^{N} \beta_n Th'_n\Big\| + \Big\|\sum_{n=2}^{N} \beta_n\big(Th'_n - (P_{s_n} - P_{s_{n-1}})Th'_n\big)\Big\| \\ &\le \|Tx\| + \max_{n\ge 2}|\beta_n| \sum_{n=2}^{N} \|Th'_n - (P_{s_n} - P_{s_{n-1}})Th'_n\| \\ &\le \|T\| + 2\sum_{n=2}^{N} \frac{\delta}{2^{n-1}} < \|T\| + 2\delta\ .\end{aligned} \tag{9.2}$$

This proves that $\widetilde{S}$ is well defined and bounded. Moreover, (9.2) implies that for each $x \in L_p(\Sigma_1)$ one has

$$\|\widetilde{S}x\| \le \big(\|T\| + 2\delta\big)\|x\|\ . \tag{9.3}$$

It remains to verify that $\widetilde{S}$ satisfies (P3). Let $x = \sum_{n=1}^{\infty} \beta_n h'_n \in L_p(\Sigma_1)$ be a mean zero sign on $[0,1]$. Using the inequality

$$|\beta_n| = \Big|\int_{[0,1]} h'^{*}_n\, x \,\mathrm{d}\mu\Big| \le \int_{[0,1]} |h'^{*}_n|\,\mathrm{d}\mu = \|h'^{*}_n\|_1 \le \|h'^{*}_n\|_q = 1\ ,$$

we obtain

$$\begin{aligned}\|\widetilde{S}x\| &= \Big\|\sum_{n=2}^{\infty} \beta_n (P_{s_n} - P_{s_{n-1}}) T h'_n \Big\| \\ &\geq \Big\|\sum_{n=2}^{\infty} \beta_n T h'_n \Big\| - \Big\|\sum_{n=2}^{\infty} \beta_n \big(T h'_n - (P_{s_n} - P_{s_{n-1}}) T h'_n\big)\Big\| \\ &\geq \|Tx\| - \sum_{n=2}^{\infty} \|T h'_n - (P_{s_n} - P_{s_{n-1}}) T h'_n\| \\ &\geq 2\delta - \sum_{n=2}^{\infty} \frac{\delta}{2^{n-1}} = 2\delta - \delta = \delta\,.\end{aligned}$$

Thus, the desired properties of $\widetilde{S}$ are proved.

Now we are ready to define S, (a_n) and (u_n). Let $V : L_p \to L_p(\Sigma_1)$ be the linear isometry extending the equality $V h_n = h'_n$ for all possible values of indices (V exists because of (P1) and (P2)). Set $S = \widetilde{S} \circ V$. Then, by (9.3), $\|S\| \leq \|\widetilde{S}\| \leq \|T\| + 2\delta$. Set $a_n = \|\widetilde{S} h'_n\|$ for all $n \in \mathbb{N}$, and $u_n = \|\widetilde{S} h'_n\|^{-1} \widetilde{S} h'_n$ if $\widetilde{S} h'_n \neq 0$, and $u_n = \|y_{s_n}\|^{-1} y_n$ if $\widetilde{S} h'_n = 0$. By (P3) and (P4), S satisfies (1) and (2) (one has $a_1 = 0$ because $S h'_1 = S h_1 = \widetilde{S} V h_1 = \widetilde{S} V h'_1 = 0$). Property (3) follows from (9.3).

If, moreover, $\|Tx\| \geq 2\delta \|x\|$ for every sign x, then $\|T h'_n\| \geq 2\delta$ for all $n \geq 2$, and by (C_n) we have

$$\begin{aligned}|a_n| = \|\widetilde{S} h_n\| &= \big\|(P_{s_n} - P_{s_{n-1}}) T h'_n \big\| \\ &\geq \|T h'_n\| - \|T h'_n - (P_{s_n} - P_{s_{n-1}}) T h'_n\| \geq 2\delta - \frac{\delta}{2^{n-1}} \geq \delta\,. \qquad \square\end{aligned}$$

Proof of Theorem 9.9 *in case* (iii). *Step 1:* $2 < p \leq r$. Suppose that an operator $T \in \mathcal{L}(L_p, \ell_r)$ is not narrow. Without loss of generality we may assume that $\|Tx\|_r \geq 2\delta$ for each mean zero sign $x \in L_p$ on $[0,1]$ and some $\delta > 0$. By Proposition 9.10 for $X = \ell_r$ and $x_n = e_n$, the unit vector basis of ℓ_r, there exists an operator $S \in \mathcal{L}(L_p, \ell_r)$ with $\|S\| \leq \|T\| + 2\delta$, which satisfies conditions (1)–(3) of Proposition 9.10. Since every normalized block basis of (e_n) is isometrically equivalent to (e_n) itself (see [79, Proposition 2.a.1]), we may and do assume that $u_n = e_n$.

Let $C > 0$ and $N \in \mathbb{N}$. We denote $I_m^k = \operatorname{supp} \overline{h}_{2^m+k} = [\frac{k-1}{2^m}, \frac{k}{2^m})$. We will define several objects depending on N and C, and in order not to complicate the notation, we omit the indices N and C. We start with the function

$$f = \frac{C}{\sqrt{N}} \sum_{n=2}^{2^{N+1}} \overline{h}_n .$$

Since S has a special form and $p \le r$, we obtain that

$$\begin{aligned}
\|Sf\|_r &= \Big\| \frac{C}{\sqrt{N}} \sum_{n=2}^{2^{N+1}} S\overline{h}_n \Big\|_r = \frac{C}{\sqrt{N}} \Big(\sum_{n=2}^{2^{N+1}} \|S\overline{h}_n\|_r^r \Big)^{1/r} \\
&\le \frac{C}{\sqrt{N}} \|S\| \Big(\sum_{m=1}^{N} \sum_{k=1}^{2^m} \|\overline{h}_{2^m+k}\|_p^r \Big)^{1/r} \\
&= \frac{C}{\sqrt{N}} (\|T\| + 2\delta) \Big(\sum_{m=1}^{N} 2^{m(1-\frac{r}{p})} \Big)^{1/r} \\
&\le C(\|T\| + 2\delta) N^{\frac{1}{r}-\frac{1}{2}}.
\end{aligned} \tag{9.4}$$

Since $r > 2$, for N large enough, $\|Sf\|_r$ is as small as we want. Our goal is to select a subset $J \subseteq \{2, \ldots, 2^{N+1}\}$ so that the element $g = \frac{C}{\sqrt{N}} \sum_{n \in J} \overline{h}_n$ is very close to a sign. This will prove that S fails (2) of Proposition 9.10, since by the special form of S, $\|Sg\|_r \le \|Sf\|_r$ which was very small.

To achieve this goal we use a technique similar to a stopping time for a martingale. A similar method was used in [59] and [53].

Set

$$A = \Big\{ \omega \in [0,1] : \max_{1 \le j \le 2^{N+1}} \Big| \frac{C}{\sqrt{N}} \sum_{i=1}^{j} \overline{h}_i(\omega) \Big| > 1 \Big\},$$

and

$$\tau(\omega) = \begin{cases} \min\Big\{ j \le 2^{N+1} : \Big| \frac{C}{\sqrt{N}} \sum_{i=1}^{j} \overline{h}_i(\omega) \Big| > 1 \Big\}, & \text{if } \omega \in A, \\ 2^N + k, \ \text{ if } \omega \notin A \ \text{ and } \ \omega \in I_N^k. \end{cases}$$

Observe that if $\tau(\omega) = 2^m + k$ then $\omega \in I_m^k$. Indeed, this is clear if $\omega \notin A$. If $\omega \in A$, then

$$\Big| \frac{C}{\sqrt{N}} \sum_{i=1}^{2^m+k-1} \overline{h}_i(\omega) \Big| \le 1,$$

so $\overline{h}_{2^m+k}(\omega) \ne 0$ and thus $\omega \in I_m^k$.

Further, if there exists $\omega \in I_m^k$ with $\tau(\omega) \ge 2^m + k$ then for every $\xi \in I_m^k$ we have $\tau(\xi) \ge 2^m + k$. Indeed, since $\omega \in I_m^k$, for every $i < 2^m + k$ and every $\xi \in I_m^k$ we have $\overline{h}_i(\omega) = \overline{h}_i(\xi)$. Thus,

$$\Big| \frac{C}{\sqrt{N}} \sum_{i=1}^{2^m+k-1} \overline{h}_i(\xi) \Big| = \Big| \frac{C}{\sqrt{N}} \sum_{i=1}^{2^m+k-1} \overline{h}_i(\omega) \Big| \le 1 .$$

Thus, $\tau(\xi) \ge 2^m + k$.

Define a set J:

$$J = \Big\{ j = 2^m + k \le 2^{N+1} : \exists \omega \in I_m^k \text{ with } \tau(\omega) \ge j \Big\}$$
$$= \Big\{ j = 2^m + k \le 2^{N+1} : \forall \xi \in I_m^k \text{ with } \tau(\xi) \ge j \Big\}.$$

Let $g : [0,1] \to \mathbb{R}$ be defined as:

$$g(\omega) = \frac{C}{\sqrt{N}} \sum_{j \le \tau(\omega)} \overline{h}_j(\omega)\,.$$

Since $\overline{h}_{2^m+k}(\omega) = 0$ for every $\omega \notin I_m^k$, we have

$$g(\omega) = \frac{C}{\sqrt{N}} \sum_{\{j=2^m+k \le \tau(\omega) : \omega \in I_m^k\}} \overline{h}_j(\omega) = \frac{C}{\sqrt{N}} \sum_{j \in J} \overline{h}_j(\omega)\,.$$

Thus, by the form of S and (9.4),

$$\begin{aligned} \|Sg\|_r &= \Big\| \frac{C}{\sqrt{N}} \sum_{j \in J} S\overline{h}_j \Big\|_r = \frac{C}{\sqrt{N}} \Big(\sum_{j \in J} \|S\overline{h}_j\|_r^r \Big)^{1/r} \\ &\le \frac{C}{\sqrt{N}} \Big(\sum_{m=1}^{N} \sum_{k=1}^{2^m} \|S\overline{h}_{2^m+k}\|_r^r \Big)^{1/r} = \|Sf\|_r \\ &\le C(\|T\| + 2\delta) N^{\frac{1}{r} - \frac{1}{2}}. \end{aligned} \tag{9.5}$$

By the definitions of $\tau(\omega)$ and $g(\omega)$, for every $\omega \in A$ one has

$$1 < |g(\omega)| < 1 + \frac{C}{\sqrt{N}}\,,$$

and for every $\omega \in [0,1] \setminus A$, $g(\omega) \neq 0$, if N is odd, being a sum of an odd number of ± 1 and zeros.

Define

$$\widetilde{g}(\omega) = \operatorname{sgn}(g(\omega)).$$

We have

$$\|g - \widetilde{g}\|_p \le \Big\| \frac{C}{\sqrt{N}} \mathbf{1}_A + \mathbf{1}_{[0,1]\setminus A} \Big\|_p.$$

Note that by the Central Limit Theorem, for large N we have

$$\begin{aligned} \mu([0,1] \setminus A) &= \mu\Big\{ \omega : \Big| \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \sum_{k=1}^{2^m} \overline{h}_{2^m+k}(\omega) \Big| \le \frac{1}{C} \Big\} \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{C}}^{\frac{1}{C}} e^{-\frac{\omega^2}{2}} d\omega \le \frac{1}{2C}. \end{aligned}$$

Thus, for large N,

$$\left\|\mathbf{1}_{[0,1]\setminus A}\right\|_p \le \Big(\frac{1}{2C}\Big)^{1/p}.$$

Hence,

$$\|g-\widetilde{g}\|_p \le \frac{C}{\sqrt{N}} + \Big(\frac{1}{2C}\Big)^{1/p}.$$

Thus, by (9.5),

$$\begin{aligned}\|S\widetilde{g}\|_r &\le \|Sg\|_r + \|S\|\|g-\widetilde{g}\|_p \\ &\le C(\|T\|+2\delta)N^{\frac{1}{r}-\frac{1}{2}} + (\|T\|+2\delta)\Big(\frac{C}{\sqrt{N}} + \Big(\frac{1}{2C}\Big)^{1/p}\Big).\end{aligned}$$

Since $r > 2$, for every $\delta > 0$, there exists $C > 0$ and N odd and large enough so that

$$\|S\widetilde{g}\|_r < \delta .$$

It remains to observe that $\widetilde{g}$ is a mean zero sign on $[0,1]$. Indeed, the support of $\widetilde{g}$ is equal to $[0,1]$, since N is odd. Observe that for every $\omega \in [0,1]$ and every $2^m + k \le 2^{N+1}$,

$$\overline{h}_{2^m+k}(\omega) = -\overline{h}_{2^m-k+1}(1-\omega) .$$

Thus, $g(\omega) = -g(1-\omega)$ for every $\omega \in [0,1]$, and

$$\mu\big(\{\omega \in [0,1] : g(\omega) > 0\}\big) = \mu\big(\{\omega \in [0,1] : g(\omega) < 0\}\big).$$

Thus, $\widetilde{g}$ is a mean zero sign on $[0,1]$, and (2) of Proposition 9.10 fails. □

Proof of Theorem 9.9 *in case* (iii). *Step 2: $r < p$.* Let $2 < r < p < \infty$ and suppose that there exists a nonnarrow operator $T \in \mathcal{L}(L_p, l_r)$. Therefore, as in Step 1, there exists an operator $S \in \mathcal{L}(L_p, \ell_r)$ with $\|S\| \le \|T\| + 2\delta$, which satisfies conditions (1)–(3) of Proposition 9.10.

For every $\nu \in \mathbb{N}$ we set

$$N_\nu = \Big\{ j = 2^m + k \ge 2 : \|S\overline{h}_j\|_r \ge \nu \cdot 2^{-m/r} \Big\} \text{ and } A_\nu = \bigcup_{2^m+k \in N_\nu} I_m^k .$$

Denote by K_ν the set of all $2^m + k \in N_\nu$ such that the interval I_m^k is maximal in $\mathcal{A} = \{I_n^i : 2^n + i \in N_\nu\}$ in the sense that it is not contained in some other interval from this system. Observe that every interval from $\mathcal{A}$ is contained in a unique maximal interval, and that distinct maximal intervals are disjoint. Hence,

$$A_\nu = \bigsqcup_{2^m+k \in K_\nu} I_m^k, \qquad \mu(A_\nu) = \sum_{2^m+k \in K_\nu} \frac{1}{2^m} .$$

Let $x_\nu = \sum_{j \in K_\nu} \overline{h}_j \in L_p$. Then $\|x_\nu\|_p \le 1$ and hence

$$\big(\|T\| + 2\delta\big)^r \ge \|Sx_\nu\|_r^r = \sum_{j \in K_\nu} \|S\overline{h}_j\|_r^r \ge \nu^r \cdot \sum_{2^m+k \in K_\nu} \frac{1}{2^m} = \nu^r \cdot \mu(A_\nu).$$

Thus, $\mu(A_\nu) \le ((3\|T\| + 2\delta)/\nu)^r$ and $\lim_{\nu\to\infty} \mu(A_\nu) = 0$.

Let $B_\nu = [0,1] \setminus A_\nu$ and $B = \bigcup_{\nu=1}^\infty B_\nu = \bigsqcup_{\nu=1}^\infty B_\nu \setminus B_{\nu-1}$, where $B_0 = \emptyset$. Then B has measure 1.

We claim that for every ν, the operator S is narrow when restricted to any of the sets $B_\nu \setminus B_{\nu-1}$. Indeed, note that on B_ν for (m,k) such that $I_m^k \cap B_\nu \ne \emptyset$ we have

$$\|S\overline{h}_{2^m+k}\|_r \le \nu \cdot 2^{-n/r} = \nu\|\overline{h}_{2^m+k}\|_r\,, \tag{9.6}$$

that is, S is bounded on the Haar system from L_r to ℓ_r. We claim that, moreover, $S\big|_{L_r(B_\nu)}$ is a bounded operator for each fixed $\nu \in \mathbb{N}$.

To see this, let $x = \sum_{n=1}^\infty \beta_n h_n^{(r)} \in L_r(B_\nu)$ where $h_{2^m+k}^{(r)} = 2^{-m/r}\overline{h}_{2^m+k}$ is the L_r-normalized Haar system. Let $(\theta_{n,k})$ be a sequence of signs, guaranteed by Lemma 7.63(2), so that

$$\Big(\sum_{n=2}^\infty |\beta_n|^r\Big)^{1/r} = \Big(\sum_{n=2}^\infty \|\beta_n h_n^{(r)}\|_r^r\Big)^{1/r} \le \Big\|\sum_{n=2}^\infty \theta_n\beta_n h_n^{(r)}\Big\|_r\,. \tag{9.7}$$

Taking into account that $S\overline{h}_1 = 0$, by (9.6) and (9.7),

$$\begin{aligned}\|Sx\|_r &= \Big(\sum_{n=2}^\infty |\beta_n|^r\|Sh_n^{(r)}\|_r^r\Big)^{1/r} \le \nu\Big(\sum_{n=2}^\infty |\beta_n|^r\Big)^{1/r} \le \nu\Big\|\sum_{n=2}^\infty \theta_n\beta_n h_n^{(r)}\Big\|_r \\ &\le \nu\Big\|\beta_1 h_1^{(r)} \sum_{n=2}^\infty \theta_n\beta_n h_n^{(r)}\Big\|_r + \nu|\beta_1| \le \nu(K_r+1)\|x\|_r\,,\end{aligned}$$

where K_r is the unconditional constant of the Haar system in L_r. Thus $S\big|_{L_r(B_\nu)}$ is bounded, and hence, by Step 1, it is narrow as an operator from $L_r(B_\nu)$ to ℓ_r. Therefore, since signs belong to both L_r and L_p, the operators $S\big|_{L_p(B_\nu)}$ and $S\big|_{L_p(B_\nu\setminus B_{\nu-1})}$ are also narrow.

Since the union of the sets $B_\nu \setminus B_{\nu-1}$ has measure 1, this yields that S is narrow, which contradicts our choice of S.

Thus, Step 2 is completed, and case (iii) of Theorem 9.9 is proved. □

Case (iii) of Theorem 9.9 can be extended to the following statement.

Corollary 9.11. *Let $2 < p, r < \infty$ and X be such that for every $n \in \mathbb{N}$, every operator $T : L_p \to \ell_r^n(X)$ is narrow. Then every operator $T : L_p \to \ell_r(X)$ is narrow.*

Corollary 9.12. *Let $2 < p, r < \infty$ and X be one of the spaces c_0, ℓ_s, with $1 \le s < \infty$, $s \ne 2$. Then every operator $T : L_p \to \ell_r(X)$ is narrow.*

Proof. This follows from Corollary 9.11 and Theorems 9.9 and 9.3, since for all $n \in \mathbb{N}$, $r > 2$, and X equal to one of the spaces c_0, ℓ_s, with $1 \le s < \infty$, $s \ne 2$, we have that $\ell_r^n(X)$ is isomorphic to X and thus every operator $T : L_p \to \ell_r(X)$ is narrow. □

By induction, we can further extend the statement of Corollary 9.12.

Corollary 9.13. *Let $n \in \mathbb{N}$, $2 < p, r_1, r_2, \ldots, r_n < \infty$ and X be one of the spaces c_0, ℓ_s, with $1 \le s < \infty$, $s \ne 2$. Then all operators in $\mathcal{L}(L_p, \ell_{r_n}(\ell_{r_{n-1}}(\ldots(\ell_{r_1}(X))\ldots)))$ are narrow.*

Our final result is that, in spite of the existence of a nonnarrow operator from L_p to ℓ_2 if $p > 2$, all these operators must be small in the following sense.

Proposition 9.14. *Let $2 < p < \infty$. Then there is no sign-embedding $T \in \mathcal{L}(L_p, \ell_2)$.*

Proof. Suppose on the contrary, that $T \in \mathcal{L}(L_p, \ell_2)$ and $\|Tx\| \ge 2\delta\|x\|$ for some $\delta > 0$ and each sign $x \in L_p$. Then by Proposition 9.10, there exists a bounded operator $S : L_p \to l_2$ which satisfies conditions (1)–(3) of Proposition 9.10. Moreover, $|a_{n,k}| \ge \delta$ for each $n = 0, 1, \ldots$ and $k = 1, \ldots, 2^n$.

Let $x_n = \sum_{k=1}^{2^n} 2^{-n/p} h_{n,k}$. Note that x_n is a mean zero sign on $[0, 1]$. Then $Sx_n = \sum_{k=1}^{2^n} 2^{-\frac{n}{p}} a_{n,k} e_{n,k}$. Now we have

$$\|Sx_n\| \ge 2^{-\frac{n}{p}} \delta 2^{\frac{n}{2}} = \delta 2^{n(\frac{1}{2}-\frac{1}{p})} .$$

Thus, $\lim_{n\to\infty} \|Sx_n\| = \infty$ which contradicts the boundedness of S. □

9.5 ℓ_2-strictly singular operators on L_p

In this section we apply methods developed in the previous section to partially answer Open problem 2.7, cf. also Open problem 7.1(b). Another, incomparable, partial answer to this problem is presented in Section 10.9.

Theorem 9.15. *For every p with $1 < p < \infty$, and every Banach space X with an unconditional basis, every ℓ_2-strictly singular operator $T : L_p \to X$ is narrow.*

The idea of the proof is very similar to the idea of Step 1 of the proof of case (iii) of Theorem 9.9. Suppose that $T : L_p \to X$ is ℓ_2-strictly singular and not narrow. As in Step 1 of the proof of case (iii) of Theorem 9.9, we use Proposition 9.10 to conclude that there exists an operator S of the particularly simple form which cannot be narrow, quantified using a specific $\delta > 0$. The ℓ_2-strict singularity of T and condition (3) of Proposition 9.10 guarantee that there exists a function x of the form

$$x = \sum_{m=1}^{N} b_m r_m = \sum_{m=1}^{N} \sum_{k=1}^{2^m} b_m \overline{h}_{2^m+k} ,$$

where (r_n) is the Rademacher system, so that $\|Sx\|_X < \delta$. Similarly as in the proof of Theorem 9.9, we construct a subset $J \subseteq \{2, \dots 2^{N+1}\}$ so that the function $g = \sum_{2^m+k \in J} b_m \overline{h}_{2^m+k}$ is very close to a sign and $\|Sg\|_X \leq \|Sf\|_X < \delta$, which gives us the desired contradiction. The only essential difference between the present proof and that of Theorem 9.9 is that in Theorem 9.9, $\|Sf\|$ was small due to the structure of the range space, and in the present proof it is due to the ℓ_2-strict singularity assumption.

Proof of Theorem 9.15. Let X be a Banach space with an unconditional basis, $1 < p < \infty$, and $T : L_p \to X$ be an ℓ_2-strictly singular operator. Suppose that an operator T is not narrow. As in the proof of Theorem 9.9, Step 1, we may assume without loss of generality that $\|Tx\|_X \geq 2\delta$ for each mean zero sign $x \in L_p$ on $[0, 1]$ and some $\delta > 0$. Therefore, there exists an operator $S \in \mathcal{L}(L_p, X)$ with $\|S\| \leq \|T\| + 2\delta$, which satisfies conditions (1)–(3) of Proposition 9.10.

Since T is ℓ_2-strictly singular, for every $N \in \mathbb{N}$, $C > 0$ and a sequence $\varepsilon_N \downarrow 0$, there exists

$$f = \sum_{m=1}^{N} b_m r_m ,$$

so that

$$\|f\|_p = 1, \quad \sum_{n=1}^{N} b_n^2 = 1 \quad \text{and} \quad \|Tf\|_X < \frac{\delta}{2C} \tag{9.8}$$

and

$$\max_{n \leq N} |b_n| \leq \varepsilon_N . \tag{9.9}$$

Indeed, to ensure that (9.9) holds, let $M \in \mathbb{N}$ with $\sqrt{M} > 2/\varepsilon$, and for $k = 1, \dots, M$, let

$$f_k = \sum_{n=m_k+1}^{m_{k+1}} b_n^{(k)} r_n ,$$

be such that $(m_k)_k$ is an increasing sequence and (9.8) holds for each k. Then

$$f = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} f_k$$

satisfies (9.8) and (9.9).

As in the proof of Theorem 9.9, Step 1, we set

$$A = \left\{\omega \in [0,1] : \max_{1\le 2^m+k\le 2^{N+1}} \left|C \sum_{2^n+i=2}^{2^m+k} b_n \overline{h}_{2^n+i}(\omega)\right| > 1\right\},$$

$$\tau(\omega) = \begin{cases} \min\left\{2^m + k \le 2^{N+1} : \left|C \sum_{2^n+i=2}^{2^m+k} b_n \overline{h}_{2^n+i}(\omega)\right| > 1\right\}, & \text{if } \omega \in A, \\ 2^N + k, \text{ if } \omega \notin A \text{ and } \omega \in I_N^k, \end{cases}$$

$$J = \left\{j = 2^m + k \le 2^{N+1} : \exists \omega \in I_m^k \text{ with } \tau(\omega) \ge j\right\}$$

and

$$g(\omega) = C \sum_{2^m+k\le\tau(\omega)} b_m \overline{h}_{2^m+k}(\omega) = C \sum_{2^m+k\in J} b_m \overline{h}_{2^m+k}(\omega).$$

Thus, by (9.9) and (c) of Proposition 9.10,

$$\begin{aligned} \|S(Cg)\|_X &\le \|T(Cg)\|_X + C\varepsilon_N\delta \\ &\le \|T(Cf)\|_X + C\varepsilon_N\delta \\ &< \frac{\delta}{2} + C\varepsilon_N\delta. \end{aligned}$$

Note that since $\varepsilon_N \downarrow 0$, by the Central Limit Theorem with the Lindeberg condition (see, e.g. [17, Theorem 27.2]), f converges to a Gaussian random variable when $N \to \infty$, so

$$\mu([0,1] \setminus A) \le \mu\left\{\omega : \left|\sum_{n=1}^{N}\sum_{i=1}^{2^n} b_n \overline{h}_{2^n+i}(\omega)\right| \le \frac{1}{C}\right\} \approx \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{C}}^{\frac{1}{C}} e^{-\frac{\omega^2}{2}} d\omega \le \frac{1}{2C}.$$

Let $[0,1] \setminus A = A_1 \sqcup A_2$, where $\mu(A_1) = \mu(A_2)$, and define

$$\widetilde{g}(\omega) = \begin{cases} \operatorname{sgn}(g(\omega)) & \text{if } \omega \in A, \\ 1 & \text{if } \omega \in A_1, \\ -1 & \text{if } \omega \in A_2. \end{cases}$$

Then $\widetilde{g}$ is a mean zero sign on $[0,1]$ and for every $\omega \in A$,

$$1 < |g(\omega)| < 1 + C\varepsilon_N.$$

Thus

$$\|Cg - C\widetilde{g}\|_p \le C\varepsilon_N + \left(\frac{1}{2C}\right)^{1/p},$$

and

$$\begin{aligned} \|S(C\widetilde{g})\|_X &\le \|S(Cg)\|_X + \|S\|C\|g - \widetilde{g}\|_p \\ &\le \left(\frac{\delta}{2} + C\varepsilon_N\delta\right) + (3\|T\| + 2\delta)\left(C\varepsilon_N + \left(\frac{1}{2C}\right)^{1/p}\right). \end{aligned}$$

Hence there exists $C > 0$ and N large enough so that

$$\|S\widetilde{g}\|_X < \delta \, ,$$

which contradicts our assumption that T is not narrow. □

Chapter 10

Narrow operators on vector lattices

As we know, every operator on an r.i. Banach space E on [0, 1] with an unconditional basis is a sum of two narrow operators (see Section 5.1). On the other hand, the sum of any two narrow operators on L_1 is narrow. This chapter is motivated by the desire to understand the reason for this discrepancy.

It is known that L_1 has "very few operators," namely all bounded operators on L_1 are regular, which is a distinctive property of L_1. This inspired O. Maslyuchenko, Mykhaylyuk and Popov to investigate narrow operators in the setting of vector lattices and to specifically study the regular narrow operators. This presented some difficulties since in vector lattices there are no analogs of characteristic functions or mean zero functions. However, in [93] (2009) O. Maslyuchenko, Mykhaylyuk and Popov found a correct extension of the notion of narrow operators to the setting of vector lattices and proved that indeed the sum of two regular narrow operators is narrow in any Banach lattice with only minor restrictions.

The goal of this chapter is to present this extension of narrowness to operators on vector lattices and to prove that the set of all narrow regular operators between two Dedekind complete vector lattices E and F, where E is atomless and F is an ideal of some order continuous Banach lattice, is a band in the lattice of all regular order continuous operators from E to F, and moreover that this band is complemented to the band generated by the lattice homomorphisms from E to F (see Theorems 10.40 and 10.41). This result both strengthens and generalizes Kalton's and Rosenthal's representation theorems for operators on L_1 to vector lattices (see Section 1.6 for details), and it extends Theorem 7.46, which was proved for operators on L_1.

The outline of the chapter is as follows: In Section 10.1 we introduce narrow and order narrow operators (Definitions 10.1 and 10.6) and we show that these two notions are different for operators from $\mathcal{L}(L_\infty)$, but that they do coincide for operators from an atomless vector lattice to an order continuous Banach lattice (Proposition 10.9). In Section 10.2 we examine whether Proposition 2.1 has an analog in vector lattices, i.e. whether every AM-compact operator is narrow. We discover that this is not the case for operators from $\mathcal{L}(L_\infty)$, but that it does hold for order-to-norm continuous operators (Theorem 10.17). Sections 10.3–10.5 build up tools for the proof of our main theorem. The general idea is similar to the proof of Theorem 1.33 and uses analogs of equivalences from Theorem 7.45. In Section 10.3 we prove that an operator T is narrow if and only if its modulus $|T|$ is narrow, strengthening a result of Flores and Ruiz on domination of narrow operators [39] (see also a survey [38]). Then we introduce a generalization of the Enflo–Starbird function λ and of λ-narrow operators, and we prove two important characterizations of λ-narrow operators (Theorems 10.30 and 10.35). In Section 10.5 we present classical theorems that we use. The main result (Theorems 10.40 and 10.41) is proved in Section 10.7. Section 10.8 is devoted to generalizations of results from Sections 10.1–10.7 to the setting of lattice-normed spaces which are a generalization of vector lattices. In Section 10.9 we present a generalization of a result of Flores and Ruiz [39] which gives a partial positive answer to Open problem 2.7, cf. also Open problem 7.1(b), for regular operators. Recall that another, incomparable, partial answer to this problem was presented in Section 9.5.

Most of the material presented here, was obtained in [93], except for Sections 10.8 and 10.9, which are based on work by Pliev [111] and Flores and Ruiz [39], respectively.

In this chapter we consider real spaces only.

10.1 Two definitions of a narrow operator on vector lattices

We introduce two notions of a narrow operator depending on whether the range space is a Banach space or a vector lattice.

Narrow operators acting from vector lattices to Banach spaces

Definition 10.1. Let E be an atomless Dedekind complete vector lattice and let X be a Banach space. A map $f : E \to X$ is called

- *narrow*, if for every $x \in E^+$ and every $\varepsilon > 0$ there exists $y \in E$ such that $|y| = x$ and $\|f(y)\| < \varepsilon$;
- *strictly narrow*, if for every $x \in E^+$ there exists $y \in E$ such that $|y| = x$ and $f(y) = 0$.

This definition can be applied also to nonlinear maps. Nevertheless, our interest in nonlinear maps will be reduced to an auxiliary result (Lemma 10.22) used to prove that AM-compact order-to-norm continuous operators are narrow. Like in the definition of a narrow operator on a Köthe space, there is no need to restrict to the atomless case in these definitions, but evidently, a narrow map must send atoms to zero.

Proposition 10.2. *Let E be a Köthe–Banach space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ) and X be a Banach space. For an operator $T \in \mathcal{L}(E, X)$ Definitions* 1.5 *and* 10.1 *of a narrow operator (resp., strictly narrow operator) are equivalent.*

Proof. Clearly, only one implication needs a proof (by Proposition 1.9, the condition $\int_\Omega x \, d\mu = 0$ can be equivalently removed from Definition 1.5). Let T satisfy Definition 1.5. Fix any $x \in E^+$ and $\varepsilon > 0$. By the absolute continuity of the norm, the linear space of simple functions is dense in E (see Proposition 2.10). Thus, there exists a simple function $u = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k}$ with $\Omega = \bigsqcup_{i=1}^{m} A_i$, $A_k \in \Sigma$ and $a_k \neq 0$ for each $k \le m$, so that $\|x-u\| < \varepsilon/(2\|T\|)$ (if $a_j = 0$ for some $j \le m$ then we can "perturb" $a_j \neq 0$ a little so the condition $\|x-u\| < \varepsilon/(2\|T\|)$ remains true; on the other hand we can assume that $T \neq 0$ because otherwise there is nothing to prove). Using Definition 1.5, we decompose $A_k = A_k' \sqcup A_k''$ so that $A_k', A_k'' \in \Sigma$ and $\|T(\mathbf{1}_{A_k'} - \mathbf{1}_{A_k''})\| < \frac{\varepsilon}{2m|a_k|}$. Let $y = x \cdot \sum_{k=1}^{m} (\mathbf{1}_{A_k'} - \mathbf{1}_{A_k''})$ and $v = \sum_{k=1}^{m} a_k (\mathbf{1}_{A_k'} - \mathbf{1}_{A_k''})$. Since $|y - v| = |x - u|$ a.e., we have $\|y - v\| = \|x - u\| < \varepsilon/(2\|T\|)$. Observe that $|y| = x$ and

$$\|Ty\| \le \|Tv\| + \|T\|\|y - v\| < \sum_{k=1}^{m} |a_k| \big\|T\big(\mathbf{1}_{A_k'} - \mathbf{1}_{A_k''}\big)\big\| + \frac{\varepsilon}{2} < \varepsilon\,.$$

The equivalence of the definitions of strictly narrow operators is proved in a similar way. □

We do not know whether the absolute continuity of the norm is essential in Proposition 10.2.

Open problem 10.3. Are Definitions 1.5 and 10.1 equivalent for every Köthe–Banach space E on a finite atomless measure space, and every Banach space X? What if $E = L_\infty$?

As explained in the introduction to this chapter, the main problem that we consider is the following.

Problem 10.4. Let E, F be Dedekind complete vector lattices with E atomless. Is the set $N_r(E, F)$ of all narrow regular operators a band in the vector lattice $L_r(E, F)$ of all regular linear operators from E to F?

It is our goal to prove that Problem 10.4 has a positive answer in most natural spaces (Theorem 10.40 below). However, in general, Problem 10.4 has a negative answer, as the following result shows.

Theorem 10.5. *The set $N_r(L_\infty)$ of all narrow regular operators on L_∞ is not a band in the vector lattice $L_r(L_\infty)$ of all regular linear operators on L_∞.*

Proof. Enumerate by $(J_n)_{n=1}^{\infty}$ the set of all intervals from $[0,1]$ with rational endpoints. Note that for any measurable set $A \subseteq [0,1]$, there exists an $n \in \mathbb{N}$ such that $\mu(J_n \setminus A) < \frac{1}{3}\,\mu(J_n)$ (we can consider any sequence $(J_n)_{n=1}^{\infty}$ with this property instead of the intervals with rational endpoints). For each $n \in \mathbb{N}$, we define an operator $T_n \in \mathcal{L}(L_\infty)$ as follows: $T_n x = \Big(\frac{1}{\mu(J_n)} \int_{J_n} x \, d\mu\Big) \mathbf{1}_{(\frac{1}{n+1}, \frac{1}{n}]}$.

Obviously, for each $n \in \mathbb{N}$ the sum $S_n = \sum_{k=1}^{n} T_k$ is a narrow positive operator. We will show that the pointwise limit (in the sense of the convergence a.e.) operator

$Tx = \lim_{n\to\infty} S_n x = \sum_{n=1}^{\infty} T_n x,\ \ x \in L_\infty$ is not narrow, while $T = \bigvee_{n=1}^{\infty} S_n$ (in fact, one can show also that $T = \bigvee_{n=1}^{\infty} T_n$). These two facts together imply that $N_r(L_\infty)$ is not a band.

Let us see that T is not narrow. Indeed, fix any $y \in L_\infty$ with $|y(t)| = 1$ a.e. on $[0,1]$. Then, at least, one of the sets $A = \{t \in [0,1] : y(t) = 1\}$ or $B = [0,1] \setminus A$ is of positive measure, say, $\mu(A) > 0$. Choose an integer n so that $\mu(J_n \setminus A) < \frac{1}{3}\,\mu(J_n)$. Then

$$\begin{aligned}\int_{J_n} y\,\mathrm{d}\mu &= \int_{J_n\cap A} y\,\mathrm{d}\mu + \int_{J_n\setminus A} y\,\mathrm{d}\mu = \mu(J_n\cap A) - \mu(J_n \setminus A)\\ &= \mu(J_n) - 2\mu(J_n\setminus A) \ge \mu(J_n) - \frac{2\mu(J_n)}{3} = \frac{\mu(J_n)}{3}\,.\end{aligned}$$

Therefore,

$$T_n y = \Big(\frac{1}{\mu(J_n)}\int_{J_n} y\,\mathrm{d}\mu\Big)\,\mathbf{1}_{\left(\frac{1}{n+1},\frac{1}{n}\right]} \ge \frac{1}{3}\,\mathbf{1}_{\left(\frac{1}{n+1},\frac{1}{n}\right]}$$

and hence, $\|Ty\| \ge \|T_n y\| \ge 1/3$.

The proof of the equality $T = \bigvee_{n=1}^{\infty} S_n$ is a standard technical exercise. Indeed, observe first that for every $n \in \mathbb{N}$ and $x \in L_\infty$ we have $S_n x = (Tx)\cdot\mathbf{1}_{(\frac{1}{n+1},1]}$. Hence, $S_n \le T$ for each n. Assume that $S_n \le S$ for each n and some $S \in L_r(L_\infty)$. Then for each n, each $x \in L_\infty^+$ and almost all $t \in (\frac{1}{n+1}, 1]$ we have $(Tx)(t) = (S_n x)(t) \le (Sx)(t)$. By arbitrariness of $n \in \mathbb{N}$, we obtain $Tx \le Sx$, and by arbitrariness of $x \in L_\infty^+$, $T \le S$. Thus, $T = \bigvee_{n=1}^{\infty} S_n$. □

Order narrow operators acting between vector lattices

Next we introduce a notion of a narrow operator for the case when the range space is a vector lattice.

Definition 10.6. Let E, F be vector lattices with E atomless. A linear operator $T : E \to F$ is called *order narrow* if for every $x \in E^+$ there exists a net (x_α) in E such that $|x_\alpha| = x$ for each α, and $Tx_\alpha \xrightarrow{\mathrm{o}} 0$.

For most cases this notion is equivalent to Definition 10.1, but in general the definitions are distinct as Example 10.8 below shows.

Proposition 10.7. *Let E be an atomless vector lattice and F be a Banach lattice. Then each narrow linear operator $T : E \to F$ is order narrow.*

Proof. If $|x_n| = x$ and $\|Tx_n\| \le 2^{-n}$ then one can show that $Tx_n \xrightarrow{\mathrm{o}} 0$. Indeed, for $z_n = \sum_{k=n}^{\infty} |Tx_k|$ we have that $|Tx_n| \le z_n \downarrow 0$. □

However, the converse is not true in general.

Example 10.8. There exists an order narrow positive operator $T \in \mathcal{L}(L_\infty)$ that is not narrow.

Proof. We show that the nonnarrow operator T constructed in the proof of Theorem 10.5 is order narrow. Fix any $x \in L_\infty^+$ and $n \in \mathbb{N}$. Define an operator $U_n : L_1 \to \ell_1^n$ by setting $U_n z = (\int_{J_1} z \cdot x \, \mathrm{d}\mu, \dots, \int_{J_n} z \cdot x \, \mathrm{d}\mu)$ for all $z \in L_1$. By Theorem 2.15, U_n is strictly narrow. So, there exists a sign y_n on $[0,1]$ so that $U_n y_n = 0$. Since $|y_n| = \mathbf{1}_{[0,1]}$, for $x_n = y_n \cdot x$ we have $|x_n| = x$ and $S_n x_n = 0$, where the operator S_n is defined in the proof of Theorem 10.5. Hence,

$$\begin{aligned} T x_n &= (T x_n) \cdot \mathbf{1}_{\left(\frac{1}{n+1}, 1\right]} + (T x_n) \cdot \mathbf{1}_{\left[0, \frac{1}{n+1}\right]} \\ &= S_n x_n + (T x_n) \cdot \mathbf{1}_{\left[0, \frac{1}{n+1}\right]} = (T x_n) \cdot \mathbf{1}_{\left[0, \frac{1}{n+1}\right]} . \end{aligned}$$

In particular, $T x_n \to 0$ a.e. on $[0,1]$. Since $(T x_n)$ is order bounded (more precisely, $-1 \le T x_n(t) \le 1$ for almost all $t \in [0,1]$), by Proposition 1.18, $T x_n \xrightarrow{\mathrm{o}} 0$. □

Nevertheless, for operators with values in order continuous Banach lattices the two notions of a narrow operator coincide. Recall that a Banach lattice E is called *order continuous* if for each net (x_α) in E the condition $x_\alpha \downarrow 0$ implies that $\|x_\alpha\| \to 0$. Note that in this case the condition $x_\alpha \xrightarrow{\mathrm{o}} 0$ also implies that $\|x_\alpha\| \to 0$.

Proposition 10.9. *Let E be an atomless vector lattice and F be an order continuous Banach lattice. Then a linear operator $T : E \to F$ is order narrow if and only if it is narrow.*

Proof. Let $T \in L(E, F)$ be order narrow. Given $x \in E^+$, let (x_α) be a net in E such that $|x_\alpha| = x$ for each α and $T x_\alpha \xrightarrow{\mathrm{o}} 0$. By the order continuity of Banach lattice F, $\|T x_\alpha\| \to 0$, and thus T is narrow. By Proposition 10.7, this ends the proof. □

A surprising fact about Definition 10.6 is that it does depend on the stated range space and it may happen that the same operator is order narrow when considered as an operator into G, but not, when considered as an operator into F for some F which contains G. Indeed, let (e_n) be any orthonormal basis of L_2 and let (h_n) be any normalized sequence of disjoint elements in L_2 such that $\|h_n\|_{L_1} \le 2^{-n}$. Consider the into isomorphism T of L_2 such that $T e_n = h_n$. As an isomorphic embedding, $T : L_2 \to L_2$ cannot be narrow (= order narrow). Nevertheless, $T : L_2 \to L_1$ is compact and hence, narrow. However this cannot happen for regular operators.

Proposition 10.10. *Let E, F, G be vector lattices such that E is atomless and F is an ideal of G. If a linear operator $T : E \to F$ is order narrow then $T : E \to G$ is order narrow as well. Conversely, if a regular linear operator $T : E \to F$ is such that $T : E \to G$ is order narrow then so is $T : E \to F$.*

Proof. The first part is trivial. Let $T : E \to F$ be a regular operator such that $T : E \to G$ is order narrow. Given any $x \in E^+$, we choose a net (x_α) in E so that $|x_\alpha| = x$ and $Tx_\alpha \xrightarrow{\text{o}} 0$, that is, $|Tx_\alpha| \le y_\alpha \downarrow 0$ for some net (y_α) in G. By regularity of T we have that $|Tx_\alpha| \le |T|\,|x_\alpha| = |T|\,x$ and hence $|Tx_\alpha| \le z_\alpha \downarrow 0$ where $|T|\,x \wedge y_\alpha = z_\alpha \in F$. Thus, $Tx_\alpha \xrightarrow{\text{o}} 0$ in F. □

We reformulate our main Problem 10.4 for order narrow operators.

Problem 10.11. Let E, F be Dedekind complete vector lattices with E atomless. Is the set $N_r^o(E, F)$ of all regular order narrow operators from E to F, a band in $L_r(E, F)$?

We remark that Theorem 10.5 does not provide a counterexample to this problem (cf. Example 10.8).

10.2 AM-compact order-to-norm continuous operators are narrow

By Proposition 2.1, every AM-compact operator $T \in \mathcal{L}(E, X)$ from a Köthe F-space E with an absolutely continuous norm on the unit to an F-space X is narrow. This is not true in general, when E is an atomless Dedekind complete vector lattice and X is a Banach space.

Example 10.12. There exists a bounded linear functional $f : L_\infty \to \mathbb{R}$ which is AM-compact but not narrow.

Proof. Denote by $\mathcal{B}$ the Boolean algebra of the Borel subsets of $[0, 1]$ equal up to a set of measure zero. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$ in the sense of [48, p. 72]. Then the linear functional $f_{\mathcal{U}} : E \to \mathbb{R}$ defined by

$$f_{\mathcal{U}}(x) = \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A x \, \mathrm{d}\mu$$

is bounded and AM-compact (a subset of L_∞ is order bounded if and only if it is norm bounded, hence, AM-compact operators defined on L_∞ are exactly compact operators). However $f_{\mathcal{U}}$ is not narrow. Indeed, for every sign $x = \mathbf{1}_A - \mathbf{1}_B$ we have $f_{\mathcal{U}}(x) = \pm 1$ depending of whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$. □

The reason why some AM-compact operators on L_∞ are not narrow is explained by the following important additional property.

Definition 10.13. Let E be a vector lattice and X be a Banach space. A map $f : E \to X$ is said to be *order-to-norm continuous* whenever it sends order convergent nets in E to norm convergent nets in X. The term f is called *order-to-norm σ-continuous* if f sends order convergent sequences in E to norm convergent sequences in X.

Observe that the functional $f_{\mathcal{U}}$ from Example 10.12 is not order-to-norm continuous. Indeed, consider a nested sequence (A_n) of members of $\mathcal{U}$ with $\mu(A_n) \to 0$. Then $\mathbf{1}_{A_n} \downarrow 0$, however $f(\mathbf{1}_{A_n}) = 1$ for each $n \in \mathbb{N}$.

For more details on order-to-norm continuous operators defined on L_∞ see Section 11.4.

Proposition 10.14. *Let E be a Banach lattice and X be a Banach space. Then every AM-compact linear operator $T : E \to X$ is bounded.*

Proof. Suppose that T is unbounded. Then there exists a sequence (x_n) in E with $\|x_n\| \leq 2^{-n}$ and $\|T x_n\| \to \infty$. Since (x_n) is order bounded by $x = \sum_{n=1}^{\infty} |x_n|$, the sequence $(T x_n)$ must be relatively compact, which is a contradiction. □

The following statement follows directly from the definitions.

Proposition 10.15. *Let E be an order continuous Banach lattice and X be a Banach space. Then every linear bounded operator $T \in \mathcal{L}(E, X)$ is order-to-norm continuous.*

Propositions 10.14 and 10.15 imply the following statement.

Corollary 10.16. *Let E be an order continuous Banach lattice and X be a Banach space. Then each AM-compact linear operator $T : E \to X$ is order-to-norm continuous.*

The main result of this section is the following theorem.

Theorem 10.17. *Let E be an atomless Dedekind complete vector lattice and X be a Banach space. Then every AM-compact order-to-norm continuous linear operator $T : E \to X$ is narrow.*

This lattice analog of Proposition 2.1 has an involved proof. First we mention a partial case of Theorem 10.17 which is of independent interest.

Corollary 10.18. *Let (Ω, Σ, μ) be an atomless measure space and X be a Banach space. Then every AM-compact order-to-norm continuous linear operator T from $L_\infty(\mu)$ to X is narrow.*

In Section 11.4 we show a different short proof of Corollary 10.18 (Theorem 11.50).

Theorem 10.17 and Corollary 10.16 imply the following statement.

Corollary 10.19. *Let E be an atomless order continuous Banach lattice and X be a Banach space. Then every AM-compact linear operator $T : E \to X$ is narrow.*

For the proof of Theorem 10.17 we need a number of lemmas, first of which is well known [52, p. 14].

Lemma 10.20 (On rounding off coefficients). *Let $(x_i)_{i=1}^n$ be a finite sequence of vectors in a finite dimensional normed space X and $(\lambda_i)_{i=1}^n$ be reals with $0 \le \lambda_i \le 1$ for each i. Then there exists a sequence $(\theta_i)_{i=1}^n$ of numbers $\theta_i \in \{0, 1\}$ such that*

$$\Big\|\sum_{i=1}^n (\lambda_i - \theta_i)\, x_i\Big\| \le \frac{\dim X}{2} \max_i \|x_i\| \,.$$

For the rest of this section, E, X and T will mean the same as in Theorem 10.17.

Lemma 10.21. *If $x \in E^+$, $|x_n| \le x$ and $x_n \perp x_m$ for all integers $n \ne m$ then* $\lim_n \|T x_n\| = 0$.

Proof. Without loss of generality, we assume that all $x_n \ge 0$. Since E is Dedekind complete, the sequence $s_n = \sum_{k=1}^n x_k = \bigvee_{k=1}^n x_k$ order converges to $s = \bigvee_{k=1}^\infty x_k$. The order-to-norm continuity of T implies that $T s_n = \sum_{k=1}^n T x_k$ converges to Ts in X. The series convergence yields that $\lim_n \|T x_n\| = 0$. □

Lemma 10.22. *The map $f : E \to \mathbb{R}$ given by $f(x) = \|T(x^+)\| - \|T(x^-)\|$ for each $x \in E$ is strictly narrow. (Note that this map is not linear.)*

Proof. Fix any $x \in E^+$. If $f(x) = 0$ then there is nothing to prove. Let $f(x) > 0$. Consider the partially ordered set $A = \{y \sqsubseteq x : f(x - 2y) \ge 0\}$ where $y_1 \le y_2$ if and only if $y_1 \sqsubseteq y_2$. If $B \subseteq A$ is a chain then $y^* = \sup B \in A$ by the order-to-norm continuity of T and, hence, of f. By the Zorn lemma, there is a maximal element $y_0 \in A$. Observe that if $y \sqsubseteq x$ then $(x - 2y)^+ = x - y$ and $(x - 2y)^- = y$. We claim that $f(x - 2y_0) = 0$. Suppose on the contrary, that $\delta = f(x - 2y_0) = \|T(x - y_0)\| - \|T y_0\| > 0$. Since E is atomless, by Lemma 10.21, there exists a further fragment $0 \ne y \sqsubseteq (x - y_0)$ with $\|Ty\| < \delta/3$. On the other hand, $y_0 + y = (y_0 \vee y) \sqsubseteq x$ and

$$\begin{aligned} f\big(x - 2(y_0 + y)\big) &= \|T(x - y - y_0)\| - \|T(y_0 + y)\| \\ &\ge \|T(x - y_0)\| - \|Ty\| - \|T y_0\| - \|Ty\| > \frac{\delta}{3}\,, \end{aligned}$$

which contradicts the maximality of y_0. □

Lemma 10.23. *Let $x \in E^+$ and (x_n) be a disjoint tree on x. If $\|T x_{2n}\| = \|T x_{2n+1}\|$ for all $n \ge 1$, then $\lim_{k\to\infty} \max_{2^k \le i < 2^{k+1}} \|T x_i\| = 0$.*

Proof. Set $\delta_k = \max_{2^k \le i < 2^{k+1}} \|T x_i\|$ and $\varepsilon = \limsup_{k\to\infty} \delta_k$. Suppose on the contrary that $\varepsilon > 0$. For each $n \in \mathbb{N}$ we set

$$\varepsilon_n = \limsup_{k\to\infty} \max_{2^k \le i < 2^{k+1}} \big\|T\big(x_i \wedge x_n\big)\big\| \,.$$

Note that if $x_i \wedge x_n > 0$ and $i > n$ then $x_i \wedge x_n = x_i$. Therefore

$$\varepsilon_n = \limsup_{k\to\infty} \max_{2^k \le i < 2^{k+1},\, x_i \sqsubseteq x_n} \|Tx_i\| .$$

Thus, for each $k \in \mathbb{N}$ we have

$$\max_{2^k \le i < 2^{k+1}} \varepsilon_i = \varepsilon . \tag{10.1}$$

We construct recursively an orthogonal subsequence $(x_{n_j})_{j=1}^{\infty}$ such that $\|Tx_{n_j}\| \ge \varepsilon/2$, that will give the contradiction with Lemma 10.21. At the first step we choose k_1 so that $\max_{2^{k_1} \le i < 2^{k_1+1}} \|Tx_i\| \ge \varepsilon/2$. By (10.1) we choose i_1, $2^{k_1} \le i_1 < 2^{k_1+1}$ so that $\varepsilon_{i_1} = \varepsilon$. Since $\|Tx_{2n}\| = \|Tx_{2n+1}\|$, we choose $n_1 \ne i_1$, $2^{k_1} \le n_1 < 2^{k_1+1}$ so that $\|Tx_{n_1}\| \ge \varepsilon/2$.

At the second step we choose $k_2 > k_1$ so that $\max_{2^{k_1} \le i < 2^{k_1+1}} \|T(x_i \wedge x_{i_1})\| \ge \varepsilon/2$. By (10.1) we choose i_2, $2^{k_2} \le i_2 < 2^{k_2+1}$ so that $\varepsilon_{i_2} = \varepsilon$. Then we choose $n_2 \ne i_2$, $2^{k_2} \le n_2 < 2^{k_2+1}$ so that $\|Tx_{n_2}\| \ge \varepsilon/2$.

Continuing this procedure, we choose the desired sequence. Indeed, $\|Tx_{n_j}\| \ge \varepsilon/2$ by the construction, and the orthogonality is guaranteed by the condition $n_j \ne i_j$, since the elements $x_{n_{j+m}}$ are components of x_{i_j} which are orthogonal to x_{n_j}. □

Lemma 10.24. *If $d = \dim X < \infty$, then T is narrow.*

Proof. Fix any $x \in E^+$ and $\varepsilon > 0$. Using Lemma 10.22, we construct recursively a disjoint tree (x_n) on x with $\|Tx_{2n}\| = \|Tx_{2n+1}\|$ for all $n \ge 1$. By Lemma 10.23, there exists k so that $d \cdot \delta_k < \varepsilon$ where $\delta_k = \max_{2^k \le i < 2^{k+1}} \|Tx_i\|$. By Lemma 10.20, there exist numbers $\theta_i \in \{0, 1\}$ for $i = 2^k, \dots, 2^{k+1} - 1$ so that

$$\Big\| \sum_{i=2^k}^{2^{k+1}-1} \Big(\frac{1}{2} - \theta_i\Big) Tx_i \Big\| \le \frac{d}{2} \max_{2^k \le i < 2^{k+1}} \|Tx_i\| = \frac{d}{2}\, \delta_k < \frac{\varepsilon}{2} .$$

Let $y = 2\sum_{i=2^k}^{2^{k+1}-1} (\frac{1}{2} - \theta_i) x_i$. Then $|y| = x$ and $\|Ty\| < \varepsilon$. □

Lemma 10.25. *For any set Γ the Banach space $E = \ell_\infty(\Gamma)$ has the approximation property, that is, for every relatively compact subset K of E and every $\varepsilon > 0$ there exists a finite rank operator $S \in \mathcal{L}(E)$ such that $\|x - Sx\| \le \varepsilon$ for each $x \in K$.*

We remark that the approximation property for classical spaces like $C(K)$ (one can consider $\ell_\infty(\Gamma)$ as $C(K)$ for a suitable compact K) was proved by Grothendieck [46]. The reader can find other proofs of the AP for $C(K)$, described in [135, p. 718]. We provide a short sketch presented to us by V. Kadets.

Sketch of proof of Lemma 10.25. Since the set of values of any element $x \in E$ belongs to the segment $[-\|x\|, \|x\|]$ (or to the closed disk of the complex plane $\mathbb{C}$ of radius $\|x\|$), which is compact, the set of all finite valued elements of E is dense in E. The main technical observation here is that every finite dimensional subspace F of E is contained in a finite dimensional subspace G of E which is 1-complemented in E and isometrically isomorphic to $\ell_\infty^{\dim G}$. Thus for every compact set K in E and every $\varepsilon > 0$ there exists a finite dimensional subspace G 1-complemented in E and such that $K \subseteq \{x \in E : (\exists g \in G)(\|x - g\| \leq \varepsilon)\}$. Then the contractive projection S from E onto G satisfies the desired property. □

Proof of Theorem 10.17. Without loss of generality, we may consider X as a subspace of some $\ell_\infty(\Gamma)$ space (we write $\subseteq$ in the sense of the obvious isometric embeddings) :

$$X \subseteq X^{**} \subseteq \ell_\infty\big(B_{X^*}\big) = \ell_\infty(\Gamma) = Z\,.$$

Fix any $x \in E^+ \setminus \{0\}$ and $\varepsilon > 0$. Since T is AM-compact, $K = \{Ty : |y| \leq x\}$ is relatively compact in X, and hence, in Z. By Lemma 10.25, there exists a finite rank operator $S \in \mathcal{L}(Z)$ such that $\|z - Sz\| \leq \varepsilon/2$ for each $z \in K$. Then $U = S \circ T : E \to Z$ is an order-to-norm continuous finite dimensional operator. Choose by Lemma 10.24 an element $y \in E$ so that $|y| = x$ and $\|Uy\| < \varepsilon/2$. Then $\|Ty\| \leq \|Uy\| + \|Ty - S\,(Ty)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

10.3 T is narrow if and only if $|T|$ is narrow

In this section we prove that the question whether a regular operator is order narrow can be reduced to the same question for a positive operator. This is an important tool for our further work.

Theorem 10.26. *Let E, F be Dedekind complete vector lattices such that E is atomless and F is an ideal of some order continuous Banach lattice. Then, every order continuous regular operator $T : E \to F$ is order narrow if and only if $|T|$ is.*

Proof. First we prove the theorem for $F = L_1(\mu)$. By Proposition 10.9, instead of order narrowness we will consider narrowness. Fix any $x \in E^+$ and $\varepsilon > 0$. Since $\{\sum_{k=1}^n |Tx_i| : x = \bigsqcup_{k=1}^n x_k,\ x_k \in E^+,\ n \in \mathbb{N}\}$ is an increasing net, by (1.4) and the order continuity of $L_1(\mu)$ there exists a finite collection $(x_k)_{k=1}^n \subset E^+$ so that

$$x = \bigsqcup_{k=1}^n x_k \quad \text{and} \quad \Big\| |T|\,x - \sum_{k=1}^n |Tx_k| \Big\| < \varepsilon\,. \tag{10.2}$$

Now we make a general remark which will be used in the proof of both implications. Let $x_k = y_k \sqcup z_k$ be any decomposition. Note that we have

$$0 \le |T|\,x - \sum_{k=1}^{n}\big(|Ty_k| + |Tz_k|\big) \le |T|\,x - \sum_{k=1}^{n} |Tx_k|\,. \tag{10.3}$$

Since $|T|\,y_k - |Ty_k|$ and $|T|\,z_k - |Tz_k|$ are positive elements of $L_1(\mu)$, the sum of their norms equals the norm of their sum. Thus, using (10.2) and (10.3), we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\Big(\big\||T|\,y_k - |Ty_k|\,\big\| + \big\||T|\,z_k - |Tz_k|\,\big\|\Big)\\ &\quad = \Big\||T|\,x - \sum_{k=1}^{n}\big(|Ty_k| + |Tz_k|\big)\Big\| \le \Big\||T|\,x - \sum_{k=1}^{n} |Tx_k|\,\Big\| < \varepsilon.\end{aligned} \tag{10.4}$$

Suppose now that T is narrow. For each $k = 1, \dots, n$, we decompose $x_k = y_k \sqcup z_k$ so that $y_k, z_k \in E^+$ and $\|Ty_k - Tz_k\| < \varepsilon/n$. Let $y = \bigsqcup_{k=1}^{n} y_k$ and $z = \bigsqcup_{k=1}^{n} z_k$. By (10.4) we get

$$\begin{aligned}\big\||T|y - |T|z\big\| &\le \sum_{k=1}^{n}\big\||T|y_k - |T|z_k\big\|\\ &\le \sum_{k=1}^{n}\big\||Ty_k| - |Tz_k|\big\| + \sum_{k=1}^{n}\Big(\big\||T|y_k - |Ty_k|\big\| + \big\||T|z_k - |Tz_k|\big\|\Big)\\ &\overset{\text{by (10.4)}}{\le} \sum_{k=1}^{n}\big\|Ty_k - Tz_k\big\| + \varepsilon < 2\varepsilon\,.\end{aligned}$$

By arbitrariness of $x \in E^+$ and $\varepsilon > 0$, this proves that $|T|$ is narrow.

Now suppose that $|T|$ is narrow. Fix any $x \in E^+$ and $\varepsilon > 0$ and let $(x_k)_{k=1}^{n} \subset E^+$ be a finite collection so that (10.2) holds. For each $k = 1, \dots, n$, we decompose $x_k = y_k \sqcup z_k$ so that

$$\big\||T|y_k - |T|z_k\big\| < \frac{\varepsilon}{n} \tag{10.5}$$

and let $y = \bigsqcup_{k=1}^{n} y_k$ and $z = \bigsqcup_{k=1}^{n} z_k$.

Since $|a - b| + |a + b| = |a| + |b| + \big||a| - |b|\big|$ for all $a, b \in \mathbb{R}$, we have that for each $u, v \in L_1(\mu)$

$$\|u - v\| = \big\||u| - |v|\big\| + \|u\| + \|v\| - \|u + v\|\,. \tag{10.6}$$

Since the L_1-norm of a sum of positive elements equals the sum of their norms, using the inequality (which follows from (1.4)) $\|\sum_{k=1}^{n}(|Ty_k| + |Tz_k|)\| \le \||T|\,x\|$, and putting $u = Ty_k$ and $v = Tz_k$ in (10.6), we obtain

$$\begin{aligned}\|Ty - Tz\| &\le \sum_{k=1}^{n} \|Ty_k - Tz_k\| \\ &= \sum_{k=1}^{n} \big\| |Ty_k| - |Tz_k| \big\| + \Big\| \sum_{k=1}^{n} \big(|Ty_k| + |Tz_k|\big) \Big\| - \Big\| \sum_{k=1}^{n} |Tx_k| \Big\| \\ &\le \sum_{k=1}^{n} \big\| |T|\, y_k - |T|\, z_k \big\| + \sum_{k=1}^{n} \Big(\big\| |T|\, y_k - |Ty_k| \big\| + \big\| |T|\, z_k - |Tz_k| \big\| \Big) \\ &+ \big\| |T|\, x \big\| - \Big\| \sum_{k=1}^{n} |Tx_k| \Big\| .\end{aligned}$$

Finally, using the above estimate, (10.5), (10.4) and the fact that

$$\big\| |T|\, x \big\| - \Big\| \sum_{k=1}^{n} |Tx_k| \Big\| = \Big\| |T|\, x - \sum_{k=1}^{n} |Tx_k| \Big\| \le \varepsilon ,$$

by (10.2), we obtain that $\|Ty - Tz\| \le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$. Since $x = y \sqcup z$, this proves that T is narrow.

Now we consider the general case. Since F is an ideal of some order continuous Banach lattice G, we have by Proposition 10.10 that $T : E \to F$ is order narrow if and only if $T : E \to G$ is, and likewise, $|T| : E \to F$ is order narrow if and only if $|T| : E \to G$ is.

We consider $T, |T| : E \to G$. Fix any $x \in E$, $x > 0$. Let E_1 and G_1 be the principal bands in E and G generated by x and $|T|\, x$, respectively. Let T_1 be the restriction of T to E_1. Evidently, $|T_1|$ coincides with the restriction of $|T|$ to E_1. Note that G_1 is an order continuous Banach lattice with the weak unit $|T| x$. Thus, by [80, Theorem 1.b.14], there exists a probability space (Ω, Σ, μ) and an ideal G_2 of $L_1(\mu)$ so that G_1 is lattice isomorphic to G_2. Let $J : G_1 \to G_2$ be a lattice isomorphism. Let $T_2 = J \circ T_1$. Obviously, $|T_2| = J \circ |T_1|$. Since J is a lattice isomorphism, the order narrowness of T_1 is equivalent to that of T_2, and the same for $|T_1|$ and $|T_2|$. By Proposition 10.10, $T_2 : E_1 \to G_2$ is order narrow if and only if $T_2 : E_1 \to L_1(\mu)$ is order narrow, and likewise, $|T_2| : E_1 \to G_2$ is order narrow if and only if $|T_2| : E_1 \to L_1(\mu)$ is order narrow. Since E_1 is an atomless Dedekind complete vector lattice, we have already proved the theorem for the operator $T_2 : E_1 \to L_1(\mu)$. Thus T_2 is order narrow if and only if $|T_2|$ is order narrow.

Suppose now that $T : E \to G$ is order narrow. Fix any $x \in E$, $x > 0$. Since T_1 (the definition of which depends on x) is order narrow as well, so is T_2 and hence $|T_2|$. Thus, there exists a net (x_α) in E_1 with $|x_\alpha| = x$ and $|T_2|\, x_\alpha \xrightarrow{\text{o}} 0$. Therefore, $|T_1|\, x_\alpha \xrightarrow{\text{o}} 0$ and hence $|T|_\alpha \xrightarrow{\text{o}} 0$. By arbitrariness of x, $|T| : E \to G$ is order narrow. Analogously, if $|T| : E \to G$ is order narrow, so is $T : E \to G$. □

Proposition 10.9 together with Theorem 10.26 imply the following result.

Corollary 10.27. *Let E be an atomless Dedekind complete vector lattice and F be an order continuous Banach lattice. Then, every order continuous regular operator $T : E \to F$ is narrow if and only if $|T|$ is narrow.*

10.4 The Enflo–Starbird function and λ-narrow operators

In this section we introduce a generalization of the Enflo–Starbird function λ and of the notion of λ-narrow operators which were studied in Section 7.2 for operators on L_1. These notions will allow us to prove a characterization of positive order narrow operators, which is an important ingredient of the proofs of our main results.

Let E be a vector lattice and $x \in E^+$. We denote by Π_x the system of all finite sets $\pi \subset E^+$ such that $x = \bigsqcup_{u \in \pi} u$. For $\pi', \pi'' \in \Pi_x$ we write $\pi' \leq \pi''$ provided for each $u \in \pi'$ there is a subset $\pi''_u \subseteq \pi''$ such that $u = \bigsqcup_{v \in \pi''_u} v$. Surely, Π_x is a directed set.

Let E, F be vector lattices with F Dedekind complete, and $T : E \to F$ be a linear operator. We define the Enflo–Starbird function $\lambda_T : E^+ \to F^+$ as follows:

$$\lambda_T(x) = \bigwedge_{\pi \in \Pi_x} \bigvee_{u \in \pi} |Tu| \,. \tag{10.7}$$

Since F is Dedekind complete, λ_T is correctly defined.

Definition 10.28. Let E, F be Dedekind complete vector lattices with E atomless. A linear operator $T : E \to F$ is called *λ-narrow* if $\lambda_T = 0$.

Here 0 denotes the zero function. Obviously, if T is regular then $\lambda_T(x) \leq \lambda_{|T|}(x)$ for each $x \in E^+$, hence if $|T|$ is λ-narrow then so is T.

The following result is an analog of Proposition 10.10 for λ-narrow operators.

Proposition 10.29. *Let E, F, G be vector lattices such that E is atomless and F is an ideal of G. A linear operator $T : E \to F$ is λ-narrow if and only if $T : E \to G$ is λ-narrow.*

Proof. Fix any $x \in E^+$ and set $\lambda_\pi = \bigvee_{u \in \pi} |Tu|$ for each $\pi \in \Pi_x$. By (10.7), $\lambda_T(x) = \inf_{\pi \in \Pi_x} \lambda_\pi$. Since F is an ideal of G, we have that $\inf_{\pi \in \Pi_x} \lambda_\pi = 0$ in F if and only if $\inf_{\pi \in \Pi_x} \lambda_\pi = 0$ in G. □

Theorem 10.30. *Let E, F be Dedekind complete vector lattices such that E is atomless and F is an ideal of some order continuous Banach lattice. Then a positive operator $T : E \to F$ is λ-narrow if and only if it is order narrow.*

Proof. By Propositions 10.10 and 10.29, without loss of generality we assume that F is an order continuous Banach lattice. Moreover, by Proposition 10.9, instead of order narrowness we will consider narrowness of T.

Suppose first that $T \geq 0$ is narrow. We show that T is λ-narrow. Fix $x \in E^+$ and $\varepsilon > 0$. It is enough to prove that there is a partition $(x_i)_1^m \in \Pi_x$ of x with $\| \bigvee_{i=1}^m Tx_i \| < \varepsilon$. First choose n so that $2^{-n}\|Tx\| < \varepsilon/2$. Using the definition of a narrow operator n times, we find a partition $(x_i)_1^{2^n} \in \Pi_x$ of x such that $\|Tx_i - 2^{-n}Tx\| < 2^{-n-1}\varepsilon$. Note that

$$\frac{Tx}{2^n} - \bigvee_{i=1}^{2^n} Tx_i \leq \frac{Tx}{2^n} - Tx_1 \leq \sum_{i=1}^{2^n} \Big| \frac{Tx}{2^n} - Tx_i \Big| . \tag{10.8}$$

On the other hand, for $i = 1, \dots, 2^n$,

$$Tx_i = \frac{Tx}{2^n} + \Big(Tx_i - \frac{Tx}{2^n} \Big) \leq \frac{Tx}{2^n} + \sum_{j=1}^{2^n} \Big| \frac{Tx}{2^n} - Tx_j \Big| .$$

Therefore

$$\bigvee_{i=1}^{2^n} Tx_i \leq \frac{Tx}{2^n} + \sum_{i=1}^{2^n} \Big| \frac{Tx}{2^n} - Tx_i \Big| ,$$

and hence

$$\bigvee_{i=1}^{2^n} Tx_i - \frac{Tx}{2^n} \leq \sum_{i=1}^{2^n} \Big| \frac{Tx}{2^n} - Tx_i \Big| .$$

Together with (10.8), this implies

$$\Big\| \bigvee_{i=1}^{2^n} Tx_i - \frac{Tx}{2^n} \Big\| \leq \sum_{i=1}^{2^n} \Big\| \frac{Tx}{2^n} - Tx_i \Big\| < \frac{\varepsilon}{2} .$$

Thus, we obtain

$$\Big\| \bigvee_{i=1}^{2^n} Tx_i \Big\| = \Big\| \bigvee_{i=1}^{2^n} Tx_i - \frac{Tx}{2^n} \Big\| + \Big\| \frac{Tx}{2^n} \Big\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

as claimed.

Suppose now that $T \geq 0$ is λ-narrow and $T \neq 0$. We shall prove that T is narrow. As in the proof of Theorem 10.26, we consider the case when $F = L_1(\mu)$ (the general case can be reduced to this one exactly like we did it in the proof of Theorem 10.26). Fix any $x \in E^+$ and $\varepsilon > 0$. By positivity of T, the net $(\bigvee_{u \in \pi} Tu)_{\pi \in \Pi_x}$ is decreasing. Thus, we can write

$$\lambda_T(x) = \lim_{\pi \in \Pi_x} \bigvee_{u \in \pi} Tu = 0 .$$

Since F is order continuous, there exists a $\pi = (x_i)_1^{2n} \in \Pi_x$ with

$$\alpha = \Big\| \bigvee_{i=1}^{2n} T x_i \Big\| \leq \frac{\varepsilon^2}{2\,\|Tx\|}\,.$$

Let $z_i = Tx_i$ for $i = 1, \dots, 2n$. Note that $K = \sum_{i=1}^{2n} \|z_i\| = \|Tx\|$. By Lemma 7.50, there exists a permutation $\tau : \{1, \dots, 2n\} \to \{1, \dots, 2n\}$ such that $\| \sum_{i=1}^{2n} (-1)^i z_{\tau(i)} \| \leq \sqrt{2\alpha K} \leq \varepsilon$. Let $y = \sum_{i=1}^{2n} (-1)^i x_{\tau(i)}$. Then $|y| = x$ and $\|Ty\| \leq \varepsilon$, and hence T is narrow. □

10.5 Classical theorems

In this section we present classical results which we will need later.

Order dense sublattices

Recall that a sublattice G of a vector lattice E is called *order dense* in E if for each $0 < x \in E$ there exists $y \in G$ with $0 < y \leq x$. Let E be a vector lattice and $e \in E^+$. We denote by $\mathfrak{C}_e$, the Boolean algebra of all components $x \sqsubseteq e$ and by $x|_e$, the band projection of an element $x \in E$ to the band $\mathrm{Band}\{e\}$ generated by e. By [6, Theorem 3.13], if $x, e \in E^+$ then $x|_e = \bigvee_{n=1}^{\infty} (x \wedge ne) \in \mathfrak{C}_e$.

Lemma 10.31. *Let E be a Dedekind complete vector lattice. Then for each $e \in E^+$ the linear subspace $G = \operatorname{span} \mathfrak{C}_e$ is an order dense sublattice of* $\mathrm{Band}\{e\}$. *In particular, $\{y \in G^+ : y \leq x\} \uparrow x$ holds for all $x \in \mathrm{Band}\{e\}^+$.*

Proof. First we prove the following claim.

> *Given any $0 < x \leq e \in E$, there exist a fragment $0 < e' \sqsubseteq e$ of e and $m \in \mathbb{N}$ such that $e' \leq mx$.*

Let $x_i = \frac{1}{i} e - x$ for $i = 1, 2, \dots$. Since E is Archimedean, $x_i \downarrow -x$ and hence $x_m^- \neq 0$ for some $m \in \mathbb{N}$. We show that the order projection given by $e' = e|_{x_m^-} = \bigvee_{n=1}^{\infty} (e \wedge n x_m^-)$ is the desired fragment of e. First we prove that $e' \neq 0$. Observe that $x_m^- \leq |x_m| \leq \frac{1}{m} e + x \leq 2e$ and $x_m^- \wedge 2e \leq 2\,(x_m^- \wedge e)$. Hence,

$$e' \geq e \wedge x_m^- \geq \frac{1}{2}\,(x_m^- \wedge 2e) \geq \frac{1}{2}\,(x_m^- \wedge x_m^-) = \frac{1}{2}\, x_m^- > 0\,.$$

Thus, $e' > 0$ is established.

Now we prove that $e' \leq mx$. To do this, we show that $e \wedge n x_m^- \leq mx$ for each $n \in \mathbb{N}$. Indeed,

$$\begin{aligned} e \wedge n x_m^- - mx &= \big(e - mx\big) \wedge \big(n x_m^- - mx\big) \\ &\leq m x_m \wedge n x_m^- \leq m x_m^+ \wedge n x_m^- = 0\,, \end{aligned}$$

which ends the proof of the claim.

Now let $0 < y \in \text{Band}\{e\}$. By [6, Remark after Theorem 3.13], $y \wedge ne \uparrow y$ and hence there exists $n \in \mathbb{N}$ such that $x = (\frac{1}{n} y) \wedge e > 0$. Since $0 < x \le e$, by the claim, there exists a fragment $0 < e' \sqsubseteq e$ and an integer m so that $\frac{1}{m} e' \le x$. This means that $\frac{n}{m} e' \le y \wedge ne \le y$. Thus, G is order dense in $\text{Band}\{e\}$. By [6, Theorem 3.1], $\{y \in G^+ : y \le x\} \uparrow x$ holds for all $x \in \text{Band}\{e\}^+$. □

Monteiro's theorem on the extension of Boolean homomorphisms

We need to extend a Boolean homomorphism which is estimated from above by a given suprema-preserving map, from a subalgebra to the entire algebra. The first theorem of this kind was obtained by Sikorski [133] but without preserving the upper estimate. The version we need is a kind of a Hahn–Banach extension theorem which is due to Monteiro [99]. There are various proofs of slightly different versions of Monteiro's theorem in the literature (see, e.g. [12, 23, 93]). In our proof we follow [93].

Definition 10.32. Let X, Y be lattices (not necessarily vector spaces). A map $\varphi : X \to Y$ is called

- $\vee$*-preserving* if $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in E$;
- $\wedge$*-preserving* if $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x, y \in E$;
- a *lattice homomorphism* provided it is both $\vee$-preserving and $\wedge$-preserving.

If, moreover, X and Y are Boolean algebras then in each of the above definitions we additionally require that $\varphi(0_X) = 0_Y$ and $\varphi(\mathbf{1}_X) = \mathbf{1}_Y$ (here $0_X = \min X$, $\mathbf{1}_X = \max X$, $0_Y = \min Y$, $\mathbf{1}_Y = \max Y$). In this case we insert the word "Boolean": a *Boolean* $\vee$*-preserving map*; a *Boolean* $\wedge$*-preserving map*; a *Boolean homomorphism*.

Theorem 10.33 (Monteiro's theorem). *Let X, Y be Boolean algebras with Y Dedekind complete, X_0 a Boolean subalgebra of X, and $\varphi : X \to Y$ be a Boolean $\vee$-preserving map. Then every Boolean homomorphism $\psi_0 : X_0 \to Y$ with $\psi_0(x) \le \varphi(x)$ for each $x \in X_0$ can be extended to a Boolean homomorphism $\psi : X \to Y$ with $\psi(x) \le \varphi(x)$ for each $x \in X$.*

Proof. First we prove that ψ_0 can be extended by one step, i.e. assume that X is the minimal Boolean algebra containing $X_0 \sqcup \{x_0\}$, where $x_0 \notin X_0$. In this case each $x \in X$ is of the form $x = (y \wedge x_0) \vee (z \setminus x_0)$ where $y, z \in X_0$. Let

$$\Pi_0 = \left\{\pi = (x_1, \dots, x_n) : n \in \mathbb{N},\ x_k \in X_0,\ \mathbf{1}_X = \bigsqcup_{i=1}^{n} x_i\right\},$$

and for each $\pi \in \Pi_0$ set $y_\pi^+ = \bigvee_{x \in \pi} \big(\psi_0(x) \wedge \varphi(x \wedge x_0)\big)$ and $y_\pi^- = \bigvee_{x \in \pi} \big(\psi_0(x) \setminus \varphi(x \setminus x_0)\big)$. We show that $y_\pi^+ \downarrow$ and $y_\pi^- \uparrow$. Indeed, let $\pi' \ge \pi$, that is, for each $x \in \pi$

there exists a subset $\pi'_x \subseteq \pi$ such that $x = \bigvee_{x' \in \pi'_x} x'$. Then

$$\begin{aligned} y^+_\pi &= \bigvee_{x\in\pi} \Big(\psi_0\big(\bigvee_{x'\in\pi'_x} x'\big) \wedge \varphi\big(\bigvee_{x'\in\pi'_x} x' \wedge x_0\big)\Big) \\ &= \bigvee_{x\in\pi}\Big(\bigvee_{x'\in\pi'_x} \psi_0(x') \wedge \bigvee_{x'\in\pi'_x} \varphi(x' \wedge x_0)\Big) \\ &\geq \bigvee_{x\in\pi} \bigvee_{x'\in\pi'_x} \psi_0(x') \wedge \varphi(x' \wedge x_0) = \bigvee_{x'\in\pi'} \big(\psi_0(x') \wedge \varphi(x' \wedge x_0)\big) = y^+_{\pi'} \end{aligned}$$

and analogously, $y^-_\pi \leq y^-_{\pi'}$. Since $\varphi(x \wedge x_0) \vee \varphi(x \setminus x_0) = \varphi((x \wedge x_0) \vee (x \setminus x_0)) = \varphi(x) \geq \psi_0(x)$ for each $x \in X_0$, we obtain that $\psi_0(x) \setminus \varphi(x \setminus x_0) \leq \psi_0(x) \wedge \varphi(x \wedge x_0)$, and hence $y^-_\pi \leq y^+_{\pi'}$. Let $y^-_\pi \uparrow y^-$ and $y^+_\pi \downarrow y^+$. Then $y^- \leq y^+$.

Let $y_0 \in Y$ be any element with $y^- \leq y_0 \leq y^+$. For each $x = (x_1 \wedge x_0) \vee (x_2 \wedge x_0)$ with $x_1, x_2 \in X_0$ we set

$$\psi(x) = \big(\psi_0(x_1) \wedge y_0\big) \vee \big(\psi_0(x_2) \setminus y_0\big).$$

We show that ψ is the desired extension. First note that if $x' \leq x_0 \leq x''$ where $x', x'' \in X_0$ then for the partitions $\pi' = \{x', x'^{\,c}\}$ and $\pi'' = \{x'', x''^{\,c}\}$ from Π we have $y^-_{\pi'} \leq y_0 \leq y^+_{\pi''}$. Therefore

$$\begin{aligned} y_0 \;\leq\; y^+_{\pi''} \;&=\; \big(\psi_0(x'') \wedge \varphi(x'' \wedge x_0)\big) \;\vee\; \big(\psi_0(x''^{\,c}) \wedge \varphi(x''^{\,c} \wedge x_0)\big) \\ &= \psi_0(x'') \;\wedge\; \varphi(x'' \wedge x_0) \;\leq\; \psi_0(x'') \end{aligned}$$

and

$$\begin{aligned} y_0 \;\geq\; y^-_{\pi'} \;&=\; \big(\psi_0(x') \setminus \varphi(x' \setminus x_0)\big) \;\vee\; \big(\psi_0(x'^{\,c}) \;\setminus\; \varphi(x'^{\,c} \setminus x_0)\big) \\ &\geq \psi_0(x') \;\setminus\; \varphi(x' \setminus x_0) \;=\; \psi_0(x'). \end{aligned}$$

Thus,

$$\bigvee_{x_0 \geq x' \in X_0} \psi_0(x') \leq y_0 \leq \bigwedge_{x_0 \leq x'' \in X_0} \psi_0(x'').$$

Using the same arguments as in the proof of Sikorski's theorem on extensions of homomorphisms [134, Section 33], we see that ψ is a homomorphism, and $\psi(x)$ does not depend on the choice of $x_1, x_2 \in X_0$ such that $x = (x_1 \wedge x_0) \vee (x_2 \setminus x_0)$. In particular, if $x \in X_0$ then

$$\psi(x) = \psi_0\big((x \wedge x_0) \vee (x \setminus x_0)\big) = \big(\psi_0(x) \wedge y_0\big) \;\vee\; \big(\psi_0(x) \setminus y_0\big) = \psi_0(x).$$

It has yet to be proved that $\psi(x) \leq \varphi(x)$ for each $x \in X$. Fix any $x_1, x_2 \in X_0$ such that $x = (x_1 \wedge x_0) \vee (x_2 \setminus x_0)$. Then $\psi(x) = (\psi_0(x_1) \wedge y_0) \vee (\psi_0(x_2) \setminus y_0)$.

Consider the partitions $\pi_i = \{x_i, x_i^c\}$, $i = 1, 2$. Since $\psi_0(x_1) \wedge \psi_0(x_1^c) = 0_X$, we obtain that

$$\begin{aligned}\psi_0(x_1) \wedge y_0 &\le \psi_0(x_1) \wedge y_{\pi_1}^+ \\ &= \psi_0(x_1) \wedge \Big(\big(\psi_0(x_1) \wedge \varphi(x_1 \wedge x_0)\big) \vee \big(\psi_0(x_1^c) \wedge \varphi(x_1^c \wedge x_0)\big)\Big) \\ &= \psi_0(x_1) \wedge \varphi(x_1 \wedge x_0) \le \varphi(x_1 \wedge x_0).\end{aligned}$$

Analogously,

$$\begin{aligned}\psi_0(x_2) \setminus y_0 &\le \psi_0(x_2) \setminus y_{\pi_2}^- \\ &= \psi_0(x_2) \setminus \Big(\big(\psi_0(x_2) \setminus \varphi(x_2 \setminus x_0)\big) \vee \big(\psi_0(x_2^c) \setminus \varphi(x_2^c \wedge x_0)\big)\Big) \\ &\le \psi_0(x_2) \setminus \Big(\psi_0(x_2) \setminus \varphi(x_2 \setminus x_0)\Big) \\ &= \psi_0(x_2) \wedge \varphi(x_2 \setminus x_0) \le \varphi(x_2 \setminus x_0).\end{aligned}$$

Thus,

$$\begin{aligned}\psi(x) &= \Big(\psi_0(x_1) \wedge y_0\Big) \vee \Big(\psi_0(x_2) \wedge y_0\Big) \le \varphi(x_1 \wedge x_0) \vee \varphi(x_2 \setminus x_0) \\ &= \varphi\big((x_1 \wedge x_0) \vee \varphi(x_2 \setminus x_0)\big) = \varphi(x).\end{aligned}$$

We now consider the general case. Let $\mathcal{H}$ be the set of all extensions $h : X_h \to Y$ such that $X_h \supseteq X_0$ is a subalgebra of X and h is a homomorphism with $h \le \varphi$. We set $h' \le h''$ if $X' \subseteq X''$ and $h''|_{X'} = h'$. Since $\mathcal{H}$ is partially ordered, by Zorn's Lemma, there exists a maximal element $\psi \in \mathcal{H}$. By the above arguments, concerning an extension by one step, $X_\psi = X$. Thus, ψ is the desired extension of ψ_0. □

10.6 Pseudonarrow operators are exactly λ-narrow operators

The following characterization is an analog of Theorem 7.45, and connects the notions of pseudonarrow (recall Definition 1.32) and λ-narrow operators.

Since we already know that an operator T is narrow if and only if $|T|$ is (see Theorem 10.26), and since the same is evidently true for pseudonarrow operators, it suffices to prove the main result of the section for positive operators only.

Theorem 10.34. *Let E, F be Dedekind complete vector lattices with E atomless. If a positive operator $T : E \to F$ is λ-narrow then it is pseudonarrow.*

Proof. Suppose that T is not pseudonarrow. Let $0 < S \le T$ be a d.p.o. and $x \in E^+$ be such that $Sx > 0$. Then for each representation $x = \bigsqcup_{k=1}^n x_k$ we have

$$Sx = \bigsqcup_{k=1}^n Sx_k = \bigvee_{k=1}^n Sx_k \le \bigvee_{k=1}^n Tx_k \,,$$

which implies that $\lambda_T(x) \ge Sx > 0$ and T is not λ-narrow. □

The converse assertion will be proved under the additional assumption of order continuity of T.

Theorem 10.35. *Let E, F be Dedekind complete vector lattices with E atomless. Then a positive order continuous operator $T : E \to F$ is λ-narrow if and only if it is pseudonarrow.*

For the proof we need a number of lemmas.

Lemma 10.36. *Let E, F be Dedekind complete vector lattices with E atomless, $T \in L^+(E, F)$ be an order continuous operator, $e \in E^+$, $f \in F^+$, and $\psi : \mathcal{C}_e \to \mathcal{C}_f$ be a Boolean homomorphism such that $\psi(x) \le Tx$ for each $x \in \mathcal{C}_e$. Then there exists a d.p.o. $S \in L^+(E, F)$ such that $S \le T$ and $Sx = \psi(x)$ for each $x \in \mathcal{C}_e$.*

Proof. For each $\pi = (x_i)_1^n \in \Pi_e$, set $L_\pi = \operatorname{span}\{x_i : i \le n\}$. Note that $\pi_1 \le \pi_2$ in Π_e implies $L_{\pi_1} \subseteq L_{\pi_2}$. In particular, $(L_\pi)_{\pi \in \Pi_e}$ is a net with respect to the inclusion.

Note that the following linear subspace is a sublattice of E

$$G = \bigcup_{\pi \in \Pi_e} L_\pi = \operatorname{span} \mathcal{C}_e \, .$$

For each $\pi = (x_i)_1^n \in \Pi_e$, define the linear operator $S_\pi : L_\pi \to F$ which extends the equality $S_\pi x_i = \psi(x_i)$ for $i = 1, \dots, n$ to L_π by linearity. Since ψ is a Boolean homomorphism on $\mathcal{C}_e$, we have that if $\pi_1 \le \pi_2$ in Π_e, then $S_{\pi_2}\big|_{L_{\pi_1}} = S_{\pi_2}$. Thus, we can define the following linear operator $\widetilde{S} : G \to F$ by setting $\widetilde{S}x = \lim_{\pi \in \Pi_e} S_\pi x$ for $x \in G$. Since $0 \le \widetilde{S} \le T$ and T is order continuous, $\widetilde{S}$ is order continuous on G.

Now we are ready to define an operator $S \in L^+(E, F)$. For $x \in E^+$ set

$$Sx = \sup\{\widetilde{S}y : y \in G \text{ and } 0 \le y \le x\} \, .$$

Sx is defined correctly since the set under the supremum is bounded above by Tx. For the same reason we also have that $Sx \le Tx$ for all $x \in E^+$.

Obviously, $Sx = \widetilde{S}x$ for each $x \in G^+$.

We prove additivity of S on E^+. Fix any $x, y \in E^+$. For each $z_1, z_2 \in G^+$ with $z_1 \le x$ and $z_2 \le y$ we have

$$S(x + y) \ge \widetilde{S}(z_1 + z_2) = \widetilde{S}z_1 + \widetilde{S}z_2 \, .$$

Passing to the supremum in the right-hand side of this inequality, since $0 \le z_1 \le x$ and $0 \le z_2 \le y$, we obtain that $S(x + y) \ge Sx + Sy$.

To prove the converse inequality, fix any $z \in G^+$ with $z \le x + y$. Since $x|_e, y|_e \in \operatorname{Band}\{e\}$, by Lemma 10.31, there exist nets $x_\alpha, y_\alpha \in G$ such that $x_\alpha \uparrow x|_e$ and $y_\alpha \uparrow y|_e$. Let $z_\alpha = z \wedge (x_\alpha + y_\alpha)$. Since $z_\alpha \le x_\alpha + y_\alpha$ and all these elements belong to the lattice G, by the Riesz decomposition property [6, Theorem 1.9], there exist nets z'_α and $z''_\alpha \in G^+$ so that $z_\alpha = z'_\alpha + z''_\alpha$, $z'_\alpha \le x_\alpha$ and $z''_\alpha \le y_\alpha$. Thus

$$\widetilde{S} z_\alpha = \widetilde{S} z'_\alpha + \widetilde{S} z''_\alpha \le \widetilde{S} x_\alpha + \widetilde{S} y_\alpha \le Sx|_e + Sy|_e \le Sx + Sy \,.$$

Since $z_\alpha \overset{\mathrm{o}}{\to} z \wedge (x+y) = z$ and $\widetilde{S}$ is order continuous on G as observed above, $\widetilde{S} z_\alpha \overset{\mathrm{o}}{\to} \widetilde{S} z$ and hence $\widetilde{S} z \le Sx + Sy$. Thus $S(x+y) \le Sx + Sy$ and the additivity of S on E^+ is proved. By the Kantorovich theorem [6, Theorem 1.7], S uniquely extends to a positive operator from E to F.

It remains to prove that S is a d.p.o. Evidently, $\widetilde{S} = S\big|_G$ is a d.p.o. Suppose, for contradiction, that there exist orthogonal $x, y \in E^+$ with $u = Sx \wedge Sy > 0$.

We claim that there exists a $z_1 \in G^+$ such that $z_1 \le x$ and $v = Sz_1 \wedge u > 0$. If not, then for each $y \in G^+$ with $y \le x$ we obtain that $Sy \wedge u = 0$. By the definition of S and distributivity (see [129, p. 52]), we have that $u = Sx \wedge u = \sup\{Sy \wedge u : y \in G \text{ and } 0 \le y \le x\} = 0$, which is a contradiction. Analogously, there exists a $z_2 \in G^+$ such that $z_2 \le y$ and $w = Sz_2 \wedge v > 0$. Thus, $Sz_1 \wedge Sz_2 \ge v \wedge Sz_2 = w > 0$. This is impossible since $S|_G$ is a d.p.o. (obviously, $z_1 \wedge z_2 \le x \wedge y = 0$). □

Definition 10.37. Let X, Y be lattices (not necessarily vector spaces). A map $f : X \to Y$ is called a *lattice antihomomorphism* if for each $u, v \in X$ we have $f(u \vee v) = f(u) \wedge f(v)$ and $f(u \wedge v) = f(u) \vee f(v)$.

Lemma 10.38. *Let E be a Dedekind complete vector lattice, $e \in E^+$. Then the formula $\mathbf{1}_e(x) = e - e|_{(e-x)^+}$ defines a lattice homomorphism $\mathbf{1}_e : E^+ \to \mathcal{C}_e$ such that $\mathbf{1}_e(x) \le x$ for all $x \in E^+$, $\mathbf{1}_e(0) = 0$ and $\mathbf{1}_e(e) = e$.*

Proof. The last two equalities are obvious. Since $e = e^+ \ge (e-x)^+$, we have that $e|_{(e-x)^+} \ge (e-x)^+|_{(e-x)^+} = (e-x)^+ \ge e - x$. Thus, $x \ge e - e|_{(e-x)^+} = \mathbf{1}_e(x)$. Since the maps $x \to x^+$ and $x \to x + e$ are lattice homomorphisms and $x \to -x$ is a lattice antihomomorphism, it remains to show that the map $0 \le x \to e|_x$ is a lattice homomorphism. By distributivity of E, one has

$$\begin{aligned} e|_{x \vee y} &= \bigvee_{n=1}^{\infty} \Big(e \wedge \big(n(x \vee y)\big)\Big) = \bigvee_{n=1}^{\infty} \Big(\big(e \wedge (nx)\big) \vee \big(e \wedge (ny)\big)\Big) \\ &= \Big(\bigvee_{n=1}^{\infty} e \wedge (nx)\Big) \vee \Big(\bigvee_{n=1}^{\infty} e \wedge (ny)\Big) = e|_x \vee e|_y . \end{aligned}$$

Since $e \wedge (nx) \uparrow e|_x$, we obtain

$$\begin{aligned} e|_{x \wedge y} &= \lim_{n\to\infty} e \wedge \big(n(x \wedge y)\big) = \lim_{n\to\infty} (e \wedge nx) \wedge (e \wedge ny) \\ &= \lim_{n\to\infty} (e \wedge nx) \wedge \lim_{n\to\infty} (e \wedge ny) = e|_x \wedge e|_y . \end{aligned}$$

□

Lemma 10.39. *Let E, F be vector lattices with F Dedekind complete. Let $T : E \to F$ be a linear operator and $e \in E^+$ be such that $f = \lambda_T(e) > 0$. Then $\varphi(x) \overset{\text{def}}{=} \mathbf{1}_f(\lambda_T(x))$ is a Boolean $\vee$-preserving map $\varphi : \mathfrak{C}_e \to \mathfrak{C}_f$ such that $\varphi(x) \le |T|x$.*

Proof. Note that $\varphi(0) = \mathbf{1}_f(\lambda_T(0)) = \mathbf{1}_f(0) = 0$ and $\varphi(e) = \mathbf{1}_f(\lambda_T(e)) = \mathbf{1}_f(f) = f$. Moreover,

$$\varphi(x) = \mathbf{1}_f\big(\lambda_T(x)\big) \le \lambda_T(x) = \bigwedge_{\pi\in\Pi_x} \bigvee_{u\in\pi} |Tu| \le \bigwedge_{\pi\in\Pi_x} \bigvee_{u\in\pi} |T|u$$
$$\le \bigwedge_{\pi\in\Pi_x} |T| \bigvee_{u\in\pi} u = \bigwedge_{\pi\in\Pi_x} |T|\,x = |T|\,x.$$

It remains to be shown that φ is a $\vee$-preserving map. First we show that $\varphi(x \sqcup y) = \varphi(x) \vee \varphi(y)$. Let $x, y \sqsubseteq e$ with $x \perp y$. Then

$$\begin{aligned}
\lambda_T(x \sqcup y) &= \lim_{\pi\in\Pi_{x\sqcup y}} \bigwedge_{\pi'\ge\pi} \bigvee_{u\in\pi'} |Tu| = \lim_{\Pi_{x\sqcup y}\ni\pi\ge\{x,y\}} \bigwedge_{\pi'\ge\pi} \bigvee_{u\in\pi'} |Tu| \\
&= \lim_{\pi\in\Pi_x,\,\tau\in\Pi_y} \bigwedge_{\pi'\ge\pi,\,\tau'\ge\tau} \bigvee_{u\in\pi'\cup\tau'} |Tu| \\
&= \lim_{\pi\in\Pi_x,\,\tau\in\Pi_y} \bigwedge_{\pi'\ge\pi,\,\tau'\ge\tau} \Big(\bigvee_{u\in\pi'} |Tu| \vee \bigvee_{v\in\tau'} |Tv|\Big) \\
&= \lim_{\pi\in\Pi_x,\,\tau\in\Pi_y} \Big(\Big(\bigwedge_{\pi'\ge\pi} \bigvee_{u\in\pi'} |Tu|\Big) \vee \Big(\bigwedge_{\tau'\ge\tau} \bigvee_{v\in\tau'} |Tv|\Big)\Big) \\
&= \lim_{\pi\in\Pi_x} \bigwedge_{\pi'\ge\pi} \bigvee_{u\in\pi'} |Tu| \vee \lim_{\tau\in\Pi_y} \bigwedge_{\tau'\ge\tau} \bigvee_{v\in\tau'} |Tv| \\
&= \bigwedge_{\pi\in\Pi_x} \bigvee_{u\in\pi} |Tu| \vee \bigwedge_{\tau\in\Pi_y} \bigvee_{v\in\tau} |Tv| = \lambda_T(x) \vee \lambda_T(y).
\end{aligned}$$

Therefore, $\varphi(x \sqcup y) = \mathbf{1}_e\big(\lambda_T(x \sqcup y)\big) = \mathbf{1}_e\big(\lambda_T(x) \vee \lambda_T(y)\big)$
$= \mathbf{1}_e\big(\lambda_T(x)\big) \vee \mathbf{1}_e\big(\lambda_T(y)\big) = \varphi(x) \vee \varphi(y).$

If $x, y \sqsubseteq e$ and $x \le y$ then $\varphi(y) = \varphi((y-x) \sqcup x) = \varphi(y-x) \vee \varphi(x) \ge \varphi(x)$. This easily implies that $\varphi(x \vee y) \ge \varphi(x) \vee \varphi(y)$ for each x, y. Moreover, $\varphi(x \vee y) = \varphi((x-y) \sqcup y) = \varphi(x-y) \vee \varphi(y) \le \varphi(x) \vee \varphi(y)$. Thus, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \sqsubseteq e$. □

Proof of Theorem 10.35. By Theorem 10.34 we only need to prove that a pseudonarrow operator is λ-narrow. Suppose that T is pseudonarrow and is not λ-narrow and $e \in E^+$ is such that $f = \lambda_T(e) > 0$. By Lemma 10.39 we construct a Boolean $\vee$-preserving map $\varphi : \mathfrak{C}_e \to \mathfrak{C}_f$ with the corresponding properties. Let

$X = \mathcal{C}_e$, $Y = \mathcal{C}_f$, $X_0 = \{0, e\}$, $\psi_0 : X_0 \to Y$ be the trivial Boolean homomorphism (i.e. $\psi_0(0) = 0$ and $\psi_0(e) = f$). Evidently, $\psi_0(x) \le \varphi(x)$ for each $x \in X_0$. By Theorem 10.33, there is a Boolean homomorphism $\psi : \mathcal{C}_e \to \mathcal{C}_f$ such that ψ extends ψ_0 with $\psi(x) \le \varphi(x)$ for all $x \in \mathcal{C}_e$. By the choice of φ, we have that $\varphi(x) \le Tx$ for all $x \in \mathcal{C}_e$. Thus, $\psi(x) \le Tx$ for all $x \in \mathcal{C}_e$. By Lemma 10.36, there exists a d.p.o. $S : E \to F$ such that $0 \le S \le T$ and $Sx = \psi(x)$ for all $x \in \mathcal{C}_e$. In particular, $Se = \psi(e) = \psi_0(e) = f > 0$ and hence $S > 0$. This means that T fails to be pseudonarrow. □

10.7 Regular narrow operators form a band in the lattice of regular operators

Using all the properties and technical tools presented in this chapter we now are able to prove the main results. First of the results gives a partial answer to Problem 10.11.

Theorem 10.40. *Let E, F be Dedekind complete vector lattices such that E is atomless and F is an ideal of some order continuous Banach lattice. Then we have the following:*

(i) *A regular order continuous operator $T : E \to F$ is order narrow if and only if it is pseudonarrow.*

(ii) *The set of all order narrow regular order continuous operators and the set of all order continuous pseudo-embeddings are mutually complemented bands in the vector lattice of all regular order continuous linear operators from E to F.*

(iii) *Each regular order continuous operator $T : E \to F$ is uniquely represented in the form $T = T_D + T_N$, where T_D is a sum of an order absolutely summable family of disjointness-preserving order continuous operators and T_N is an order continuous order narrow operator. Moreover,* $\max\{\|T_D\|, \|T_N\|\} \le \|T\|$.

Proof. Note that by [80, p. 7], F is Dedekind complete. We prove (i) by the following scheme

$$T \text{ is order narrow} \overset{\text{Theorem 10.26}}{\iff} |T| \text{ is order narrow} \overset{\text{Theorem 10.30}}{\iff} |T| \text{ is } \lambda\text{-narrow}$$

$$\overset{\text{Theorem 10.35}}{\iff} |T| \text{ is pseudonarrow} \overset{\text{by definition}}{\iff} T \text{ is pseudonarrow}.$$

Items (ii) and (iii) follow from (i) by Corollary 1.34. □

If E and F are order continuous Banach lattices then the order continuity of T in Theorem 10.40 can be removed because in this case every regular operator is order continuous. Thus, by Proposition 10.9, our main result on Banach lattices can be formulated as follows.

Theorem 10.41. *Let E, F be order continuous Banach lattices with E atomless. Then we have the following:*

(i) *A regular operator $T : E \to F$ is narrow if and only if it is pseudonarrow.*

(ii) *The set $N_r(E, F)$ of all narrow regular operators and the set $L_{pe}(E, F)$ of all pseudo-embeddings are mutually complemented bands in the vector lattice $L_r(E, F)$ of all regular linear operators from E to F.*

(iii) *Each regular operator $T : E \to F$ is uniquely represented in the form $T = T_D + T_N$, where T_D is a sum of an order absolutely summable family of disjointness-preserving operators and T_N is narrow. Moreover,* $\max\{\|T_D\|, \|T_N\|\} \le \|T\|$.

We do not know whether the assumption of order continuity of the operators in Theorem 10.40 can be omitted.

Open problem 10.42. Is Theorem 10.40 true for regular operators, which are not order continuous?

In particular, what happens for $E = L_\infty$? The following question was posed in [93] (cf. also Open problem 11.62).

Open problem 10.43. Is the set of all order narrow regular operators $T : L_\infty \to L_\infty$ a band in the vector lattice $L_r(L_\infty)$ of all regular linear operators on L_∞?

Lattice versions of some problems on the isomorphic structure of L_p

One of the most interesting easy to formulate problems in Banach space theory that is still unsolved, is to characterize, up to an isomorphism, infinite dimensional complemented subspaces of L_1. In particular, is it true that they have to be isomorphic to either ℓ_1 or L_1? An attentive consideration of papers devoted to the geometry of the space L_1 of the last 30 years confirms that many authors were really motivated by this problem in their investigations.

A lattice approach to narrow operators explained why in L_1 the sum of two narrow operators is narrow, and why this is not the case for operators on L_p with $1 < p < \infty$. The reason is that there are "few" operators on L_1, all of them regular. In contrast, there are many operators on L_p for $1 < p < \infty$. In particular, nonregular ones that make many subspaces complemented by means of nonregular projections. The same reason explains why the Haar system is unconditional in L_p but not in L_1, and many other properties that distinguish L_p from L_1. This has led the authors of [93] to define the following notion that makes it possible to generalize the complemented subspaces problem for L_1 to the setting of the spaces L_p, with $1 \le p < \infty$.

Definition 10.44. Let E be a Banach lattice. A subspace F of E is called *regularly complemented* if there exists a regular projection from E onto F.

Open problem 10.45. Let $1 \leq p < \infty$, $p \neq 2$. Is every regularly complemented subspace of L_p isomorphic to either ℓ_p or L_p?

The following lattice version of Open problem 7.52 was posed in [93].

Open problem 10.46. Let E be an order continuous Banach lattice and $T \in L_r(E)$ be a regular operator. Suppose that for each band $F \subseteq E$ the restriction $T|_F$ is not an isomorphic embedding. Must T be narrow?

Theorem 7.30 gives an affirmative answer for $E = L_1$.

10.8 Narrow operators on lattice-normed spaces

There is a generalization of narrow operators to the domain space which is a generalization of a vector lattice. New domain space is called a lattice-normed space, and was introduced by Kantorovich in 1936 and later developed by a number of mathematicians, see Kusraev's book [74]. In [111] Pliev adopted the notion of a narrow operator to the setting of operators defined on a lattice-normed space and valued in a Banach space, and to order narrow operators acting between lattice-normed spaces. All results in this section are from [111], with preliminaries from [74].

Definition of a lattice-normed space

As before, all vector lattices are assumed to be Archimedean. Let V be a vector space and E be a vector lattice. A map $\lvert\!\lvert \cdot \rvert\!\rvert : V \to E^+$ is called a *vector norm* if the following conditions hold for all $x, y \in V$ and $\lambda \in \mathbb{R}$:

(1) $\lvert\!\lvert x \rvert\!\rvert = 0 \quad \Leftrightarrow \quad x = 0$;

(2) $\lvert\!\lvert \lambda x \rvert\!\rvert = |\lambda| \lvert\!\lvert x \rvert\!\rvert$;

(3) $\lvert\!\lvert x + y \rvert\!\rvert \leq \lvert\!\lvert x \rvert\!\rvert + \lvert\!\lvert y \rvert\!\rvert$.

If these conditions hold, the triple $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ is called a *lattice-normed space*. The vector lattice E is called the norm lattice of the vector norm $\lvert\!\lvert \cdot \rvert\!\rvert$. We will also denote a lattice-normed space $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ by (V, E); this is especially convenient if one considers two lattice-normed spaces, because in this case we denote both vector norms by the same symbol $\lvert\!\lvert \cdot \rvert\!\rvert$ (as we do for norms on different normed spaces), and then understand which vector norm we mean in each case by the context.

A vector norm $\lvert\!\lvert \cdot \rvert\!\rvert$ is called *decomposable* if

(4) for each $x \in V$ and $e_1, e_2 \in E^+$ with $\lvert\!\lvert x \rvert\!\rvert = e_1 + e_2$ there are $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $\lvert\!\lvert x_k \rvert\!\rvert = e_k$ for $k = 1, 2$.

A lattice-normed space $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ is called *decomposable* if its vector norm is decomposable.

Examples

(a) The special case when $V = E$ with $\lvert x \rvert = |x|$ for each $x \in V$ shows that the notion of a lattice-normed space generalizes the notion of a vector lattice, and its decomposition property follows from Riesz decomposition property for vector lattices [6, Theorem 1.9].

(b) Let V be a normed space and $E = \mathbb{R}$ with the vector norm equal to the scalar norm $\lvert x \rvert = \|x\|$ for each $x \in V$. The decomposition property trivially holds.

(c) Let (Ω, Σ, μ) be a finite measure space, E an order dense ideal in $L_0(\mu)$, and X a Banach space. Let $L_0(\mu, X)$ be the linear space of all equivalence classes of strongly measurable functions $x : \Omega \to X$. For any $x \in L_0(\mu, X)$, we define an element $\lvert x \rvert \in L_0(\mu)$ by $\lvert x \rvert(\omega) = \|x(\omega)\|$ for almost all $\omega \in \Omega$. Define $E(X) = \{x \in L_0(\mu, X) : \lvert x \rvert \in E\}$. Then $E(X) = (E(X), \lvert \cdot \rvert, E)$ is a lattice-normed space having the decomposition property [74, Lemma 2.3.7]. Moreover, if E is a Banach lattice then the lattice-normed space $E(X)$ is a Banach space with respect to the norm $|||x||| = \| \|x(\cdot)\|_X \|_E$.

A space with a mixed norm

Let E be a Banach lattice and let (V, E) be a lattice-normed space. Since $\lvert x \rvert \in E$ for each $x \in V$, we can set

$$|||x||| = \big\| \lvert x \rvert \big\|, \quad \text{for each } x \in V . \tag{10.9}$$

Then $||| \cdot |||$ is a norm on V and the normed space $(V, ||| \cdot |||)$ is called a *space with a mixed norm*, or a *Banach space with a mixed norm* if it is complete with respect to this norm.

In view of the inequality $|\lvert x \rvert - \lvert y \rvert| \le \lvert x - y \rvert$ and monotonicity of the norm in E, we have $\| \lvert x \rvert - \lvert y \rvert \| \le |||x - y|||$ for all $x, y \in E$, so a vector norm is a norm continuous map from $(V, ||| \cdot |||)$ to E.

Different notions on a lattice-normed space

Notions of order convergence, order boundedness, Dedekind (order) completeness, order ideal, disjoint elements, fragment, atom, atomlessness, disjoint tree, positive operator, dominated operator, regular operator, modulus of an operator, and many related ones are defined in a similar way as for ordered vector spaces and vector lattices. Note that for the order convergence in a lattice-normed space the following notation is used: $x_\alpha \overset{\text{bo}}{\to} x$. More precisely, let (V, E) be a lattice-normed space. Then $x_\alpha \overset{\text{bo}}{\to} x$ for a net (x_α) in V and $x \in V$ means that $\lvert x_\alpha - x \rvert \overset{\text{o}}{\to} 0$ in E. Here (bo) stands for "Banach order."

There is one more difference. It is not hard to show that a vector lattice is Dedekind complete (that is, every order bounded from above nonempty set has the least upper bound) if and only if it is order complete (i.e. every order Cauchy net order converges). However a lattice-normed space need not have the lattice property that every two-point set has the least upper bound. Thus the two possible notions of completeness in a lattice-normed space are not equivalent to each other. We say that a lattice-normed space V is *(bo)-complete* if every (bo)-Cauchy net in V is (bo)-convergent in V. A decomposable (bo)-complete lattice-normed space is called a *Banach–Kantorovich space*.

Let V be a lattice-normed space. A subspace V_0 of V is called *(bo)-ideal* of V if for each $x \in V$ and $y \in V_0$ the inequality $|x| \leq |y|$ implies $x \in V_0$. A subspace V_0 of V is a (bo)-ideal in V if and only if $V_0 = \{x \in V : \lvert\!\lvert x \rvert\!\rvert \in L\}$ for some ideal L of E [74, Item 2.1.6.(1)]. As in vector lattices, two elements $x, y \in V$ are called *disjoint* (write $x \perp y$) if $\lvert\!\lvert x \rvert\!\rvert \wedge \lvert\!\lvert y \rvert\!\rvert = 0$, that is, $\lvert\!\lvert x \rvert\!\rvert \perp \lvert\!\lvert y \rvert\!\rvert$ in E. Given a subset $M \subseteq V$, we set $M^d = \{x \in V : (\forall y \in M)(x \perp y)\}$. An element $z \in V$ is called a *fragment* $x \in V$ (write $z \sqsubseteq x$) provided $0 \leq \lvert\!\lvert z \rvert\!\rvert \leq \lvert\!\lvert x \rvert\!\rvert$ and $z \perp (x - z)$ (cf. definition of a fragment on p.13). Two fragments x_1, x_2 of x are called *mutually complemented* if $x = x_1 + x_2$ (*MC fragments*, in short). As in vector lattices, the equality $v = \bigsqcup_{k=1}^n v_k$ means that $v = \sum_{k=1}^n v_k$ and $v_i \perp v_j$ if $i \neq j$.

Narrow operators

Definition 10.47. Let (V, E) be a lattice-normed space with an atomless vector lattice E, and X be a Banach space. A linear operator $T : V \to X$ is called *(bo)-narrow* if for every $v \in V$ there are two mutually complemented fragments v_1 and v_2 of v such that $\|T(v_1 - v_2)\| < \varepsilon$. If, moreover, for each $v \in V$ there are two MC fragments v_1 and v_2 of v such that $\|T(v_1 - v_2)\| = 0$ then T is called *(bo)-strictly narrow*.

The following statement shows that for a lattice-normed space $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ which coincides with $E = (E, |\cdot|, E)$, Definitions 10.1 and 10.47 are equivalent.

Proposition 10.48. *Let E be an atomless Dedekind complete vector lattice and X be a Banach space. A linear operator $T : E \to X$ is narrow (resp., strictly narrow) in the sense of Definition* 10.1 *if and only if it is (bo)-narrow (resp., (bo)-strictly narrow), being considered as an operator defined on the lattice-normed space $E = (E, |\cdot|, E)$ in the sense of Definition* 10.47.

Proof. If T is (bo)-narrow (resp., (bo)-strictly narrow) then T is obviously narrow (resp., strictly narrow) in the sense of Definition 10.1.

Let T be narrow in the sense of Definition 10.1. Fix any $v \in E$ and $\varepsilon > 0$, and write $v = v^+ - v^-$. By Definition 10.1, there exist y_1 and y_2 so that $|y_1| = v^+$, $|y_2| = v^-$, $\|Ty_1\| < \varepsilon/2$ and $\|Ty_2\| < \varepsilon/2$. Set $v_1 = y_1^+ - y_2^-$ and $v_2 = y_1^- - y_2^+$.

Then $v_1 \perp v_2$ and $v = v^+ - v^- = |y_1| - |y_2| = y_1^+ + y_1^- - y_2^+ - y_2^- = v_1 + v_2$, that is, v_1 and v_2 are MC fragments of v. Moreover,

$$\|T(v_1 - v_2)\| = \|T(y_1 + y_2)\| \le \|Ty_1\| + \|Ty_2\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Dominated operators

Definition 10.49. Let (V, E) and (W, F) be lattice-normed spaces, $T \in L(V, W)$ a linear operator and $S \in L^+(E, F)$. We say that S *dominates* or *majorizes* T, and that S is a *dominant* or *majorant* for T, if $\lvert\!\lvert Tv \rvert\!\rvert \le S \lvert\!\lvert v \rvert\!\rvert$ for each $v \in V$.

The set of all dominants of T is denoted by $\operatorname{maj} T$. If there is the least element of $\operatorname{maj} T$ in $L_r(E, F)$ then it is called the *least* or the *exact dominant* of T and is denoted by $|T|$. The set of all dominated operators from V to W is denoted by $M(V, F)$. Denote by E_0^+ the conic hull of the set $\lvert\!\lvert V \rvert\!\rvert = \{\lvert\!\lvert v \rvert\!\rvert : v \in V\}$, that is, the set of all elements of E of the form $\sum_{k=1}^n \lvert\!\lvert v_k \rvert\!\rvert$ where $n \in \mathbb{N}$; $v_1, \dots, v_n \in V$. If E has the principal projection property, that is, each band in E generated by one element is a projection band, then $E_0^+ = \lvert\!\lvert V \rvert\!\rvert$ [74, Theorem 4.1.4]. By Lemma 1.24, this holds for a Dedekind complete vector lattice E. Note that, on the other hand, E_0^+ is the positive cone of the band $E_0 = \lvert\!\lvert V \rvert\!\rvert^{dd}$ in E.

Let us present some auxiliary known statements useful for beginners.

Proposition 10.50 ([74, Theorem 2.1.2(2)]). *Let (V, E) be a lattice-normed space. If $v_1, v_2 \in V$ are disjoint elements then $\lvert\!\lvert v_1 + v_2 \rvert\!\rvert = \lvert\!\lvert v_1 \rvert\!\rvert + \lvert\!\lvert v_2 \rvert\!\rvert$.*

Proposition 10.51 ([74, Theorem 2.1.2(3)]). *Let (V, E) be a lattice-normed space. If $e_1, e_2 \in E$ are disjoint elements then for each $v \in V$ the decomposition $v = v_1 + v_2$ with $\lvert\!\lvert v_1 \rvert\!\rvert = e_1$ and $\lvert\!\lvert v_2 \rvert\!\rvert = e_2$ is unique.*

Proposition 10.52 ([74, Theorem 4.1.2]). *Let (V, E) and (W, F) be lattice-normed spaces with (V, E) decomposable and F Dedekind complete. Then every dominated operator $T \in L(V, W)$ has the exact dominant $\lvert\!\lvert T \rvert\!\rvert$.*

Proposition 10.53 ([74, Theorem 4.1.5]). *Let (V, E) and (W, F) be lattice-normed spaces and let an operator $T \in L(V, W)$ have the exact dominant $\lvert\!\lvert T \rvert\!\rvert$. Then*

(a) $\forall e \in E_0^+ \quad \lvert\!\lvert T \rvert\!\rvert e = \sup\Big\{\sum_{k=1}^n \lvert\!\lvert Tv_k \rvert\!\rvert : n \in \mathbb{N},\ v_k \in V,\ \sum_{k=1}^n \lvert\!\lvert v_k \rvert\!\rvert = e\Big\}$;

(b) $\forall e \in E^+ \quad \lvert\!\lvert T \rvert\!\rvert e = \sup\{\lvert\!\lvert T \rvert\!\rvert e_0 : e_0 \in E_0^+,\ e_0 \le e\}$;

(c) $\forall e \in E \quad \lvert\!\lvert T \rvert\!\rvert e = \lvert\!\lvert T \rvert\!\rvert e^+ - \lvert\!\lvert T \rvert\!\rvert e^-$.

Order narrow operators

Definition 10.54. Let (V, E) and (W, F) be lattice-normed spaces with atomless E. A linear operator $T : V \to W$ is called *(bo)-order narrow* if for every $v \in V$ there is a net (v'_α, v''_α) of pairs of MC fragments of v such that $T(v'_\alpha - v''_\alpha) \xrightarrow{\text{bo}} 0$.

The following two statements are analogs of Propositions 10.7 and 10.9.

Proposition 10.55. *Let (V, E) be a lattice-normed space with an atomless vector lattice E and let (W, F) be a space with a mixed norm. Then every (bo)-narrow operator $T : V \to W$ is (bo)-order narrow.*

We omit the proof which is the same as the proof of Proposition 10.7.

Proposition 10.56. *Let (V, E) be a lattice-normed space with an atomless vector lattice E and let (W, F) be a space with a mixed norm where F is a Banach lattice with an order continuous norm. Then a linear operator $T : V \to W$ is (bo)-narrow if and only if it is (bo)-order narrow.*

Proof. By Proposition 10.55, we need to prove only one implication. Suppose that T is (bo)-order narrow. Fix any $v \in V$ and $\varepsilon > 0$. Then there exists a net (v'_α, v''_α) of pairs of MC fragments of v such that $T(u_\alpha) \xrightarrow{\text{bo}} 0$, where $u_\alpha = v'_\alpha - v''_\alpha$. The last condition means by definition that $|\!|Tu_\alpha|\!| \xrightarrow{\text{o}} 0$. By the order continuity of the norm in F, $|||Tu_\alpha||| = \| |\!|Tu_\alpha|\!| \| \to 0$. Thus, there is α with $|||Tu_\alpha||| < \varepsilon$. □

We will use also the following auxiliary lemma.

Lemma 10.57. *Let (V, E), (J, F_1) and (W, F) be lattice-normed spaces, E an atomless vector lattice, J a (bo)-ideal of W and F_1 an order ideal of F. If a dominated linear operator $T : V \to J$ is (bo)-order narrow then so is $T : V \to W$. Conversely, for a dominated linear operator $T : V \to J$, if $T : V \to W$ is (bo)-order narrow then so is $T : V \to J$.*

Proof. The first part is obvious. Let $T : V \to J$ be a dominated operator with $T : V \to W$ (bo)-order narrow. For any $v \in V$ there exists a net (v_α) in V such that every element v_α is a difference $u_1 - u_2$ of two MC fragments of u with $Tv_\alpha \xrightarrow{\text{bo}} 0$, that is, $|Tv_\alpha| \le y_\alpha \downarrow 0$ for some net (y_α) in F. Since $|\!|T|\!| : E \to F_1$, we have that $|\!|Tv_\alpha|\!| \le |\!|T|\!||\!|v|\!| = g \in F_1$. Hence, $|\!|Tv_\alpha|\!| \le z_\alpha \downarrow 0$ where $z\alpha = g \wedge y_\alpha$ is a net in F_1. Thus, the net (Tv_α) (bo)-converges to 0 in (J, F_1). □

GAM-compact operators

We introduce analogs of order-to-norm continuous and AM-compact operators for the new setting.

Definition 10.58. Let (V, E) be a lattice-normed space and X be a Banach space. A linear operator $T : V \to X$ is called:

- *(bo)-to-norm continuous* if it sends (bo)-convergent nets from V to convergent nets in X;
- *generalized AM-compact* or *GAM-compact* if it sends (bo)-bounded sets from V to relatively compact sets in X.

Observe that if $V = E$ is a vector lattice then Definition 10.58 gives exactly order-to-norm continuous and AM-compact operators.

The following theorem is a generalized version of Theorem 10.17.

Theorem 10.59. *Let (V, E) be a Banach–Kantorovich space, where E is an atomless Dedekind complete vector lattice, and X be a Banach space. Then every GAM-compact (bo)-to-norm continuous operator $T : V \to X$ is (bo)-narrow.*

We omit the proof of Theorem 10.59 which repeats the proof of Theorem 10.17 with minor adjustments (the reader can find the proof in [111]).

Main results

The following results are due to Pliev [111].

Theorem 10.60. *Let E, F be Dedekind complete vector lattices with E atomless and F an ideal of an order continuous Banach lattice, and (V, E) be a Banach–Kantorovich space. Then every (bo)-continuous dominated linear operator $T : V \to F$ is (bo)-order narrow if and only if $\lvert\!\lvert T \rvert\!\rvert : E \to F$ is order narrow.*

Proof. As in the proof of Theorem 10.26, we first prove the theorem for $F = L_1(\mu)$. By Propositions 10.56 and 10.9, we equivalently replace order narrowness with narrowness. Fix any $e \in E_0^+$ and $\varepsilon > 0$. Since E is Dedekind complete, $E_0^+ = \lvert\!\lvert V \rvert\!\rvert$ by the discussion after Definition 10.49. Let $v \in V$ be such that $\lvert\!\lvert v \rvert\!\rvert = e$. Let Π_e be the system of all finite sets $\pi \subset V$ such that $v = \sum_{u \in \pi} u$ and $e = \bigsqcup_{u \in \pi} \lvert\!\lvert u \rvert\!\rvert$. For $\pi', \pi'' \in \Pi_x$ we write $\pi' \le \pi''$, if for each $u \in \pi'$ there is a subset $\pi''_u \subseteq \pi''$ such that $u = \sum_{w \in \pi''_u} w$ and $\lvert\!\lvert u \rvert\!\rvert = \bigsqcup_{w \in \pi''_u} \lvert\!\lvert w \rvert\!\rvert$. By decomposability of (V, E) and Proposition 10.51, Π_e is a directed set. Consider the net $\big(\sum_{u \in \pi} |Tu|\big)_{\pi \in \Pi_e}$ in $L_1(\mu)$. By the triangle inequality, this net increases. By Proposition 10.53(a),

$$\begin{aligned}\lvert\!\lvert T \rvert\!\rvert e &= \sup\Big\{\sum_{k=1}^{n} |Tv_k| : n \in \mathbb{N}, v_k \in V, \sum_{k=1}^{n} \lvert\!\lvert v_k \rvert\!\rvert = e\Big\} \\ &= \sup\Big\{\sum_{u \in \pi} |Tu| : \pi \in \Pi_e\Big\}.\end{aligned}$$

By the order continuity of $L_1(\mu)$, there is $\pi \in \Pi_e$ such that $\| |T| e - \sum_{u\in\pi} |Tu| \| < \varepsilon$. Let $\pi = (v_k)_{k=1}^n$. We have that

$$v = \sum_{k=1}^n v_k, \ e = \bigsqcup_{k=1}^n |v_k| \ \text{and} \ \Big\| |T| e - \sum_{k=1}^n |Tv_k| \Big\| < \varepsilon . \tag{10.10}$$

Now we make a general remark. Let $v_k = u_k \sqcup w_k$, for each $k = 1, \dots, n$, be a decomposition into MC fragments. Then setting $u = \bigsqcup_{k=1}^n u_k$ and $w = \bigsqcup_{k=1}^n w_k$, we obtain that $v = u + w$, and

$$e = \bigsqcup_{k=1}^n |v_k| = \bigsqcup_{k=1}^n |u_k| \sqcup \bigsqcup_{k=1}^n |w_k| \overset{\text{byProposition 10.50}}{=} |u| + |w| ,$$

is a decomposition into MC fragments. Hence

$$|T| e = \sum_{k=1}^n \big(|T||u_k| + |T||w_k| \big) . \tag{10.11}$$

By Proposition 10.53(a),

$$0 \le |T| e - \sum_{k=1}^n \big(|Tu_k| + |Tw_k| \big) \le |T| e - \sum_{k=1}^n |Tv_k| . \tag{10.12}$$

Since $|T||u_k| - |Tu_k|$ and $|T||w_k| - |Tw_k|$ are positive elements of $L_1(\mu)$, the sum of their norms equals the norm of the sum. Thus,

$$\sum_{k=1}^n \Big(\big\| |T||u_k| - |Tu_k| \big\| + \big\| |T||w_k| - |Tw_k| \big\| \Big) \tag{10.13}$$

$$\overset{\text{by (10.11)}}{=} \Big\| |T| e - \sum_{k=1}^n \big(|Tu_k| + |Tw_k| \big) \Big\| \overset{\text{by (10.12)}}{\le} \Big\| |T| e - \sum_{k=1}^n |Tv_k| \Big\| \overset{\text{by (10.10)}}{<} \varepsilon.$$

Suppose first that T is (bo)-narrow. Then, for each $k = 1, \dots, n$, there exist MC fragments u_k and w_k of v_k so that $\|Tu_k - Tw_k\| < \varepsilon/n$. Thus, as observed above, $u = \bigsqcup_{k=1}^n u_k$ and $w = \bigsqcup_{k=1}^n w_k$ are MC fragments of v. Hence

$$\begin{aligned}
\big\| |T|\,|u| - |T|\,|w| \big\| &\le \sum_{k=1}^n \big\| |T|\,|u_k| - |T|\,|w_k| \big\| \\
&\le \sum_{k=1}^n \big\| |Tu_k| - |Tw_k| \big\| + \sum_{k=1}^n \Big(\big\| |T|\,|u_k| - |Tu_k| \big\| + \big\| |T|\,|w_k| - |Tw_k| \big\| \Big) \\
&\overset{\text{by (10.13)}}{\le} \sum_{k=1}^n \big\| |Tu_k| - |Tw_k| \big\| + \varepsilon \le \sum_{k=1}^n \big\| Tu_k - Tw_k \big\| + \varepsilon < 2\varepsilon .
\end{aligned}$$

By arbitrariness of $e \in E_0^+$ and $\varepsilon > 0$, $\mathbf{|}T\mathbf{|}$ is narrow.

Suppose now that $\mathbf{|}T\mathbf{|}$ is narrow. Fix any $v \in V$ and $\varepsilon > 0$. Let $e = \mathbf{|}v\mathbf{|}$. By the order continuity of $L_1(\mu)$, there exists $\pi = (v_k)_{k=1}^n \in \Pi_e$ so that (10.10) holds. By narrowness of $\mathbf{|}T\mathbf{|}$, we decompose $v_k = u_k \sqcup w_k$ so that $\|\mathbf{|}T\mathbf{|}\,\mathbf{|}u_k\mathbf{|} - \mathbf{|}T\mathbf{|}\,\mathbf{|}w_k\mathbf{|}\| < \varepsilon/n$. Setting $u = \bigsqcup_{k=1}^n u_k$ and $w = \bigsqcup_{k=1}^n w_k$, we obtain by Proposition 10.53(a),

$$\Big\| \sum_{k=1}^n \big(|Tu_k| + |Tw_k|\big) \Big\| \le \big\| \mathbf{|}T\mathbf{|}e \big\| . \tag{10.14}$$

By (10.10),

$$\big\| \mathbf{|}T\mathbf{|}\,e \big\| - \Big\| \sum_{k=1}^n |Tv_k| \Big\| = \Big\| \mathbf{|}T\mathbf{|}\,e - \sum_{k=1}^n |Tv_k| \Big\| \le \varepsilon . \tag{10.15}$$

Since the norm of the sum of positive elements in $L_1(\mu)$ equals the sum of their norms,

$$\begin{aligned}
\|Tu - Tw\| &\le \sum_{k=1}^n \|Tu_k - Tw_k\| \\
&\overset{\text{by (10.4)}}{=} \sum_{k=1}^n \big\| |Tu_k| - |Tw_k| \big\| + \Big\| \sum_{k=1}^n \big(|Tu_k| + |Tw_k|\big) \Big\| - \Big\| \sum_{k=1}^n |Tv_k| \Big\| \\
&\overset{\text{by (10.14)}}{\le} \sum_{k=1}^n \big\| \mathbf{|}T\mathbf{|}\mathbf{|}u_k\mathbf{|} - \mathbf{|}T\mathbf{|}\mathbf{|}w_k\mathbf{|} \big\| + \sum_{k=1}^n \Big(\big\| \mathbf{|}T\mathbf{|}\mathbf{|}u_k\mathbf{|} - |Tu_k| \big\| \\
&\qquad + \big\| \mathbf{|}T\mathbf{|}\mathbf{|}w_k\mathbf{|} - |Tw_k| \big\| \Big) + \big\| \mathbf{|}T\mathbf{|}e \big\| - \Big\| \sum_{k=1}^n |Tv_k| \Big\| \\
&\overset{\text{by (10.13)}}{\le} 2\varepsilon + \big\| \mathbf{|}T\mathbf{|}e \big\| - \Big\| \sum_{k=1}^n |Tv_k| \Big\| \overset{\text{by (10.15)}}{\le} 3\varepsilon.
\end{aligned}$$

By arbitrariness of $v \in V$ and $\varepsilon > 0$, T is (bo)-narrow.

Now we consider the general case. Since F is an ideal of an order continuous Banach lattice H, we have by Lemma 10.57 that $T : V \to F$ is (bo)-order narrow if and only if $T : V \to H$ is. Consider $T : V \to H$ and $\mathbf{|}T\mathbf{|} : E \to H$. Fix any $v \in V$. Let E_1 and H_1 be the principal bands in E and H generated by $\mathbf{|}v\mathbf{|}$ and $\mathbf{|}T\mathbf{|}\mathbf{|}v\mathbf{|}$, respectively. By [74, 2.1.2 (1)], there exists a Boolean isomorphism $h : \mathcal{B}(E_0) \to \mathcal{B}((V, E))$ between the Boolean algebras of all bands in E_0 and (V, E) (the assumption of [74, 2.1.2 (1)] is satisfied since $E_0^+ = \mathbf{|}V\mathbf{|}$). Let $V_1 = h(E_1)$ and T_1 be the restriction $T|_{V_1}$ of T to V_1. Since V_1 and E_1 are bands, the operator $\mathbf{|}T_1\mathbf{|} : E_1 \to H$ coincides with the restriction $\mathbf{|}T\mathbf{|}|_{E_1}$ of $\mathbf{|}T\mathbf{|}$ to E_1. Since H_1 is an order continuous Banach lattice with the weak unit $\mathbf{|}T\mathbf{|}\mathbf{|}v\mathbf{|}$, by [80, Theorem 1.b.14], there exist a probability space (Ω, Σ, μ) and an ideal H_2 of $L_1(\mu)$ such that H_1 is isomorphic to H_2. Let $S : H_1 \to H_2$ be a lattice isomorphism and $T_2 = S \circ T_1$. Observe that $|T_2| = S \circ |T_1|$. Since S is a lattice isomorphism, the (bo)-order narrowness of T_1 is equivalent to that of T_2, and the same is true for $|T_1|$ and $|T_2|$. By our previous considerations, $T_2 : V_1 \to H_2$ is (bo)-order narrow if and only if $|T_2| : E_1 \to H_2$ is order narrow.

Let $T : V \to H$ be (bo)-order narrow. Fix any $v \in V$. Since T_1 (the definition of which does depend on v) is (bo)-order narrow, so is T_2 and hence $|T_2|$ is order narrow. So there exists a net (v_α) in V_1 such that every element v_α is a difference $u_1 - u_2$ of two MC fragments of v with $|T_2||v_\alpha| \xrightarrow{o} 0$. Therefore, $|T||v_\alpha| = |T_1||v_\alpha| \xrightarrow{o} 0$. By arbitrariness of $v \in V$, $|T| : E \to H$ is order narrow. Analogously, if $|T| : E \to H$ is order narrow then $T : V \to H$ is (bo)-order narrow. □

Let $(V, E), (W, F)$ be lattice-normed spaces and let $M(V, W)$ be the space of all dominated operators from V to W. Let $L_{pe}(E, F)$ be the set of all pseudo-embeddings from E to F and set $L_{pe}(V, W) = \{T \in M(V, W) : |T| \in L^+_{pe}(E, F)\}$.

Theorem 10.61. *Let E, F be order complete vector lattices with E atomless, F an ideal of some order continuous Banach lattice, and (V, E) be a Banach–Kantorovich space. Then every (bo)-continuous dominated linear operator $T : V \to F$ is uniquely represented in the form $T = T_{pe} + T_{on}$, where $T_{pe} \in L_{pe}(V, W)$ and T_{on} is a (bo)-continuous (bo)-order narrow operator.*

Proof. By [74, 4.2.1], the set $M(V, E)$ is a lattice-normed space with the vector norm $p : M(V, E) \to L^+(E, F)$ defined by $p(T) = |T|$ for every $T \in \mathcal{M}(V, E)$. By [74, 4.2.6], p is decomposable. This means that for every dominated operator $T : V \to F$ its exact dominant $|T| : E \to F$ has the following property:

$$\begin{aligned} &\text{if } |T| = S_1 + S_2 \text{ then there are } T_1, T_2 \in \mathcal{M}(V, F) \text{ such that} \\ &|T_1| = S_1,\ |T_2| = S_2,\ 0 \le S_1, S_2 \text{ and } S_1 \perp S_2. \end{aligned} \tag{10.16}$$

Fix an arbitrary (bo)-continuous dominated operator $T : V \to F$. By [74, 4.3.2], every dominated operator $T : V \to F$ is (bo)-continuous if and only if $|T| : E \to F$ is order continuous. Hence, $|T|$ is order continuous. Thus by Theorem 10.40, the positive order continuous operator $|T|$ is uniquely represented as a sum of order continuous operators $|T| = S_D + S_N$, where $S_D \in L_{pe}(E, F)$ and S_N is an order narrow operator. Since $|T|$ is positive and $S_D \perp S_N$, we have that S_D and S_N are also positive. By (10.16) we obtain that there exist $T_{pe}, T_{on} \in \mathcal{M}(V, F)$ such that $|T_{pe}| = S_D$ and $|T_{on}| = S_N$. By Theorem 10.60, the proof is completed. □

10.9 ℓ_2-strictly singular regular operators are narrow

In this section we present a generalization of a result of Flores and Ruiz [39] which gives a partial positive answer to Open problem 2.7, cf. also Open problem 7.1(b), for regular operators. Another, incomparable, partial answer to this problem is presented in Section 9.5.

Recall that a Banach lattice E is called *q-concave*, for $1 \le q < \infty$, if there exists $M > 0$ such that for every $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in E$ we have

$$\Big(\sum_{k=1}^{n} \|x_k\|^q\Big)^{1/q} \le M \Big\|\Big(\sum_{k=1}^{n} |x_k|^q\Big)^{1/q}\Big\| .$$

The following theorem is the main result of the section.

Theorem 10.62. *Let E be a Köthe–Banach space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ) such that E is a q-concave Banach lattice for some $1 \le q < \infty$, and let F be an order continuous Banach lattice. Then every regular ℓ_2-strictly singular operator $T \in L_r(E, F)$ is narrow.*

Let E be a Köthe–Banach space (Ω, Σ, μ). Given a set $A \in \Sigma$ and a sub-σ-algebra Σ_1 of Σ, by $E(A, \Sigma_1)$ we denote the subspace of $E(A)$ consisting of all Σ_1-measurable functions.

The following lemma is a generalized version of a result of M. Kadets-Pełczyński [50]. The idea of the proof below is taken from [80, Proposition 1.c.8].

Lemma 10.63. *Let E be a Köthe–Banach space on a finite measure space (Ω, Σ, μ) with an absolutely continuous norm. Let (x_n) be an order bounded sequence from E so that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $x_n \notin M_\varepsilon^E$, where*

$$M_\varepsilon^E = \big\{x \in E : \mu\big\{t \in \Omega : |x(t)| \ge \varepsilon \|x\|_E\big\} \ge \varepsilon\big\} .$$

Then there exist a subsequence (y_n) of (x_n) and a disjoint sequence (z_n) in E such that $|z_n| \le |y_n|$ for all n, and $\|y_n - z_n\| \to 0$.

Proof. Let $e \in E^+$ be such that $|x_n| \le e$ for all $n \in \mathbb{N}$. Choose a subsequence (x_n') of (x_n) so that $x_n' \notin M_{2^{-n}}^E$ for all n. For every $n \in \mathbb{N}$, let $A_n = \{t \in \Omega : |x_n'(t)| \ge 2^{-n}\|x_n'\|\}$ and $B_n = \bigcup_{k=n}^{\infty} A_k$.

Note that $\mu(A_n) < 2^{-n}$, $B_{n+1} \subseteq B_n$ and $\mu(B_n) \le 2^{-n+1}$ for each n. Let $(n_i)_i$ be a strictly increasing sequence of integers so that $\|e \cdot \mathbf{1}_{B_{n_{i+1}}}\| \le 1/i$.

Observe that the sets $C_i = A_{n_i} \setminus B_{n_{i+1}}$ are disjoint. Let $y_i = x_{n_i}'$ and $z_i = y_i \cdot \mathbf{1}_{C_i}$ for $i = 1, 2, \dots$. Then (z_i) is a disjoint sequence, $|z_i| \le |y_i|$, and

$$\begin{aligned}\|y_i - z_i\| &= \|x_{n_i}' \cdot \mathbf{1}_{\Omega \setminus C_i}\| \le \|x_{n_i}' \cdot \mathbf{1}_{\Omega \setminus A_{n_i}}\| + \|x_{n_i}' \cdot \mathbf{1}_{B_{n_{i+1}}}\| \\ &\le \|2^{-n_i}\|x_{n_i}'\| \cdot \mathbf{1}_{\Omega \setminus A_{n_i}}\| + \|e \cdot \mathbf{1}_{B_{n_{i+1}}}\| \\ &\le 2^{-n_i}\|e\|\|\mathbf{1}_\Omega\| + 1/i \to 0 \text{ as } i \to \infty .\end{aligned}$$

□

Theorem 10.62 follows from the next result.

Theorem 10.64. *Let E, F be Köthe–Banach spaces with absolutely continuous norms on finite measure spaces $(\Omega_E, \Sigma_E, \mu_E)$ and $(\Omega_F, \Sigma_F, \mu_F)$, respectively, the first of which is atomless. Let $T \in L_r(E, F)$ be a nonnarrow regular operator. Then there exist $A \in \Sigma_E^+$ and a separable atomless sub-σ-algebra $\widetilde{\Sigma}$ of $\Sigma_E(A)$ such that the restriction $T|_{E(A,\widetilde{\Sigma})}$ is L_1-to-L_1 bounded, and its continuous extension $\widetilde{T} : L_1(A, \widetilde{\Sigma}) \to L_1(\mu_F)$ is an isomorphic embedding.*

Proof of Theorem 10.64. Our first observation is that, by Proposition 1.26, E and F are σ-order continuous Banach lattices. Further, without loss of generality we may assume that both measure spaces are [0, 1] with the Lebesgue measure.

Indeed, since T is nonnarrow, there exist $C \in \Sigma_E^+$ and $\eta > 0$ such that $\|Tx\| \geq \eta$ for every sign x on C. Let Σ_1 be any separable atomless sub-σ-algebra of $\Sigma_E(C)$ and $E_1 = E(C, \Sigma_1)$. Let $T_1 = T|_{E_1}$ and $F_1 = \overline{T(E(C, \Sigma_1))}$. Since F_1 is a separable subspace of F, the sub-σ-algebra Σ_2 of Σ_F generated by F_1 is also separable. Thus, $T_1 : E_1 \to F_1$ is a nonnarrow operator. By the Carathéodory theorem, the measure spaces $(C, \Sigma_1, \mu_E|_{\Sigma_1})$ and $([0,1], \Sigma, \mu)$ are isomorphic, and the measure space $(\Omega_F, \Sigma_2, \mu_F|_{\Sigma_2})$ is isomorphic to $([0,1], \Sigma_3, \mu)$ for a suitable sub-σ-algebra Σ_3 of Σ. Let $\tau_1 : C \to [0,1]$ and $\tau_2 : \Omega_F \to [0,1]$ be measure preserving, up to constant multiples, maps that generate isomorphisms of the corresponding measure spaces. Define Köthe–Banach spaces E_2 and F_2 on [0, 1] as the ranges of the maps $J_1 : E_1 \to L_1$ and $J_2 : F_1 \to L_1$ defined by $(J_1 x)(t) = x(\tau_1^{-1}(t))$, for $x \in E_1$, and $(J_2 y)(t) = y(\tau_2^{-1}(t))$, for $y \in F_1$, endowed with the norms $\|J_1 x\|_{E_2} = \|x\|_{E_1}$ and $\|J_2 y\|_{F_2} = \|y\|_{F_1}$, respectively. Then the operator $T_2 : E_2 \to F_2$ defined by $T_2 x = J_2(T_1(J^{-1}x))$ is nonnarrow, and the spaces E_2, F_2 are Köthe–Banach spaces on [0, 1] that are σ-order continuous Banach lattices. Moreover, E_2 and F_2 have absolutely continuous norms. Thus, if the theorem holds for the case when both measure spaces are [0, 1], then the operator T_2, and hence, T have the desired properties.

Our goal is to find $B \in \Sigma^+$ such that the restriction $T|_{E(B)}$ is L_1-to-L_1 bounded. By Corollary 2.11, E is separable, and so is the subspace $Z = \overline{T(E)}$ of F. By Lemma 1.28, there is an ideal X of F with a weak unit and containing Z. By Lemma 1.29, there exists a probability space (Ω', Σ', μ') so that X is order isometric to some Banach lattice $(Y, \|\cdot\|_Y)$ which is an ideal of $L_1(\mu')$, and such that (b)–(d) of Lemma 1.29 hold.

For simplicity of the notation, we assume that $Y = X \subseteq L_1(\mu')$ and $\|\cdot\|_X = \|\cdot\|_Y$ on X. By our convention, $T, T^+, T^- : E \to X \subseteq L_1(\mu')$. By Proposition 1.9(iii), there exist $A' \in \Sigma^+$ and $\delta \in (0, \mu(A')/4)$ such that $\|Tx\| \geq \delta$ for each sign x with $\operatorname{supp} x = B' \subseteq A'$ and $\mu(B') \geq \mu(A')/2$. Since $\mathbf{1}_{\Omega'} \in X' = X^*$, we have that $(T^+)^*\mathbf{1}_{\Omega'}, (T^-)^*\mathbf{1}_{\Omega'} \in E^* = E' \subseteq L_1$. Let $A_1 \in \Sigma(A')$ with $\mu(A' \setminus A_1) < \delta$ and $M_1 > 0$ so that $\mathbf{1}_{A_1} \cdot (T^+)^*\mathbf{1}_{\Omega'} \leq M_1 \cdot \mathbf{1}_{[0,1]}$. Then there exist $B \in \Sigma(A_1)$ with $\mu(A_1 \setminus B) < \delta$ and $M_2 > 0$ so that $\mathbf{1}_B \cdot (T^-)^*\mathbf{1}_{\Omega'} \leq M_2 \cdot \mathbf{1}_{[0,1]}$. We claim that B

has the desired property. Indeed, given any $x \in E(B)^+$, we have

$$\begin{aligned}\|T^+x\|_1 &= \int_{\Omega'} T^+x \, d\mu' = \langle \mathbf{1}_{\Omega'}, T^+x \rangle = \langle (T^+)^* \mathbf{1}_{\Omega'}, x \rangle = \int_{[0,1]} (T^+)^* \mathbf{1}_{\Omega'} \cdot x \, d\mu \\ &= \int_{[0,1]} \mathbf{1}_{A_1} \cdot (T^+)^* \mathbf{1}_{\Omega'} \cdot x \, d\mu \le \int_{[0,1]} M_1 \cdot \mathbf{1}_{[0,1]} \cdot x \, d\mu = M_1 \|x\|_1 \, .\end{aligned}$$

Analogously, $\|T^-x\|_1 \le M_2\|x\|_1$ for all x^+, and hence, $T|_{E(B)}$ is L_1-to-L_1 bounded.

Since $E(B)$ is dense in $L_1(B)$, there is a continuous extension $\widetilde{T}' : L_1(B) \to L_1$. And since $\delta < \mu(A')/4$, we have

$$\mu(B) = \mu(A') - \mu(A' \setminus A_1) - \mu(A_1 \setminus B) > \mu(A') - 2\delta > \mu(A') - \frac{\mu(A')}{2} = \frac{\mu(A')}{2} .$$

Hence, by the choice of A' and δ, $\|Tx\| \ge \delta$ for each sign x on B.

Our next goal is to show that, by regularity of T, $\|Tx\|_1 \ge \delta_1$ for some $\delta_1 > 0$ and every sign x on B. Assuming the contrary, we choose a sequence (v_n) of signs on B with $\|Tv_n\|_1 \to 0$. If $Tv_n \in M_\varepsilon^X$ for all n and some $\varepsilon > 0$, then $\|Tv_n\|_1 \ge \varepsilon^2 \|Tv_n\|_X \ge \varepsilon^2\delta$, for all n, which is a contradiction. Thus, we obtain that for every $\varepsilon > 0$ there is n so that $Tv_n \notin M_\varepsilon^X$. Since the sequence (v_n) is order bounded and T is regular, the sequence (Tv_n), and hence, (z_n) is order bounded as well. By Lemma 10.63, there exists a disjoint sequence (z_n) in X so that $|z_n| \le |Tv_n|$, for all n and $\|z_n - Tv_n\|_X \to 0$. By the Fremlin–Meyer-Nieberg theorem [6, Theorem 12.13, p. 183], $\|z_n\|_X \to 0$. This is impossible, since $\|z_n\|_X \ge \|Tv_n\|_X - \|z_n - Tv_n\|_X \ge \delta - \|z_n - Tv_n\|_X \to \delta$.

Thus, $\|Tx\|_1 \ge \delta_1$ for some $\delta_1 > 0$ and every sign x on B. Hence, $\widetilde{T}' : L_1(B) \to L_1$ is nonnarrow. By Theorem 7.30, there exists $A \in \Sigma(B)^+$ such that the restriction $\widetilde{T} = \widetilde{T}'|_{L_1(A)}$ is an isomorphic embedding. As a restriction of $T|_{E(B)}$ to $E(A)$, $T|_{E(A)}$ is L_1-to-L_1 bounded and has the continuous extension $\widetilde{T} : L_1(A) \to L_1$. □

Proof of Theorem 10.62. Let $T \in L_r(E, F)$ be nonnarrow. By the definition of a narrow operator, there exist $C \in \Sigma^+$ and a separable atomless sub-σ-algebra Σ_1 of $\Sigma(C)$ such that the restriction $T_1 = T|_{E_1}$ is nonnarrow where $E_1 = E(C, \Sigma_1)$. By Corollary 2.11, E_1 is separable, and so is the subspace $Z = \overline{T(E_1)}$ of F. By Lemma 1.28, there is an ideal X of F containing Z and having a weak unit. By Lemma 1.29, there exists a probability space (Ω', Σ', μ') such that X is order isometric to a Banach lattice $(Y, \|\cdot\|_Y)$ which is an ideal of $L_1(\mu')$. For simplicity of the notation, we assume that $Y = X \subseteq L_1(\mu')$ and $\|\cdot\|_X = \|\cdot\|_Y$ on X. By Theorem 10.64, there exist $A \in \Sigma_1^+$ and an atomless sub-σ-algebra $\widetilde{\Sigma}$ of $\Sigma_1(A)$ such that the restriction $T_2|_{E_1(A,\widetilde{\Sigma})}$ is L_1-to-L_1 bounded, and its continuous extension $\widetilde{T} : L_1(A, \widetilde{\Sigma}) \to L_1(\mu')$ is an isomorphic embedding. Let (r_n) be a Rademacher system in $L_1(A, \widetilde{\Sigma})$ and $R = [r_n]$ its closed linear span in $L_1(A, \widetilde{\Sigma})$. Since E

is q-concave for some $q < \infty$, by the generalized Khintchine inequality [80, Theorem 1.d.6(i)], R is closed in E and is isomorphic to ℓ_2. Then for each $x \in R$ we have

$$\|Tx\|_F \geq \|Tx\|_1 \geq \|\widetilde{T}^{-1}\|^{-1}\|x\|_1 \geq \|\widetilde{T}^{-1}\|^{-1}C^{-1}\|x\|_E \ ,$$

where C is the constant from the generalized Khintchine inequality. □

Chapter 11

Some variants of the notion of narrow operators

In this chapter we consider a few variants of the notion of narrow operators and their applications.

In Section 11.1 we present hereditarily narrow operators which were introduced by V. Kadets, Kalton and Werner in [53] (2005). These are narrow operators with the additional property that they remain narrow when restricted to certain subspaces of the domain (see Definition 11.1). The biggest difference between this property and the usual narrowness is that the class of hereditarily narrow operators is closed under addition. Using this notion, V. Kadets, Kalton and Werner [53] proved that L_1 does not sign-embed in any Banach space with an unconditional basis (Corollary 11.12) which generalizes a result of Pełczyński concerning isomorphic embeddings.

In Section 11.2 we introduce, following [102], a notion weaker than narrowness, where instead of requiring an existence of a $\{1, -1, 0\}$-valued function x so that $\|Tx\| < \varepsilon$, we have a weaker condition saying that x does not grow too rapidly (see Definition 11.19). We prove that for operators on L_p, $1 < p \le 2$, the gentle narrow condition implies that the operator is narrow in the usual sense (Theorem 11.20). This partially answers Open problem 7.52 which asks whether it is possible to drop any restrictions on an element x, except for the support of x, in the definition of narrowness.

In Section 11.3 we present C-narrow operators, introduced by V. Kadets and Popov [57] in 1996, which extend the notion of narrow operators to operators defined on $C(K)$-spaces. This approach is based on the idea of Rosenthal's characterization of narrow operators on L_1 (Theorem 7.30). A different approach, based on the Daugavet property, was introduced by V. Kadets, Shvidkoy and Werner [63] in 2001. We discuss it briefly in the introduction to Section 11.3.

The last two sections are devoted to the usual notion of narrow operators but in somewhat unusual settings. Majority of results in the theory of narrow operators use the absolute continuity of the norm of the domain space. This assumption fails for L_∞, so the study of narrow operators on the norm of L_∞ requires different techniques and many surprising results are true in this setting. For example, continuous linear functionals do not need to be narrow (Examples 10.12 and 11.46). In Chapter 10 we showed a few initial theorems about narrow operators defined on L_∞, but the full account of known results and open problems in this setting is presented in Section 11.4. In Section 11.5 we prove that every 2-homogeneous scalar polynomial on L_p, $1 \le p < 2$ is narrow. We are not aware of any other results concerning narrowness of polynomials on Banach spaces, however we believe that they will appear in the future.

11.1 Hereditarily narrow operators

Definition and first properties

As we know, the class of a narrow operators in general is not closed under addition or restrictions of operators to subspaces of the domain space (see Corollary 4.16). V. Kadets, Kalton and Werner in [53] introduced a variant of the notion of narrow operators, which has good ideal properties and is closed under certain restrictions of the domain; they called this new notion hereditarily narrow operators.

Definition 11.1. Let E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ), and let X be a Banach space. An operator $T \in \mathcal{L}(E, X)$ is called *hereditarily narrow* if for every $A \in \Sigma^+$ and every atomless sub-σ-algebra $\mathcal{F}$ of $\Sigma(A)$ the restriction of T to $E(\mathcal{F})$ is narrow (here $E(\mathcal{F}) = \{x \in E(A) : x \text{ is } \mathcal{F} - \text{measurable}\}$).

Clearly, each hereditarily narrow operator is narrow, however, the converse is not true (see Corollary 4.16).

Our first result is that hereditarily narrow operators behave very well under addition.

Proposition 11.2 ([53]). *Let E be a Köthe–Banach space on* $[0, 1]$ *with an absolutely continuous norm, and X be a Banach space. Then the sum $T = T_1 + T_2$ of a narrow operator $T_1 \in \mathcal{L}(E, X)$ and a hereditarily narrow operator $T_2 \in \mathcal{L}(E, X)$ is narrow. In particular, the sum of two hereditarily narrow operators is hereditarily narrow.*

Proof. We fix any $A \in \Sigma^+$ and $\varepsilon > 0$. By Proposition 2.19 we choose an atomless sub-σ-algebra Σ_1 of $\Sigma(A)$ such that $\|T_1\| \le \varepsilon/2$ where $T_1 = T|_{E^0(\Sigma_1)}$, where $E^0(\Sigma_1) = \{x \in E(\Sigma_1) : \int_{[0,1]} x \, d\mu = 0\}$. Since T_2 is hereditarily narrow, we can choose a mean zero sign $x \in E(\Sigma_1)$ with $\|T_2 x\| < \varepsilon/2$. Then x is a sign on A and $\|Tx\| \le \|T_1 x\| + \|T_2 x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

The following direct consequence of the definition, along with earlier results, will give us many examples of hereditarily narrow operators.

Proposition 11.3. *Let X be a Banach space, and E be a Köthe–Banach space on a finite atomless measure space (Ω, Σ, μ) such that for every $A \in \Sigma$ and every atomless sub-σ-algebra $\mathcal{F}$ of $\Sigma(A)$, the conditional expectation operator $M^{\mathcal{F}} : E \to E(\mathcal{F})$ is well defined and bounded. If every operator $T \in \mathcal{L}(E, X)$ is narrow then every operator $T \in \mathcal{L}(E, X)$ is hereditarily narrow.*

Thus, if there exists a class of operators closed under restrictions, whose members are all narrow operators, then they are hereditarily narrow. In particular, we have the following.

Corollary 11.4. *Operators from the following classes are hereditarily narrow:*

- *every compact or AM-compact operator $T \in \mathcal{L}(E, X)$, for any Banach space X and any r.i. Banach space E with an absolutely continuous norm on the unit on a finite atomless measure space (cf. Proposition* 2.1*);*
- *every Dunford–Pettis operator $T \in \mathcal{L}(E, X)$, for any Banach space X and any r.i. Banach space E with an absolutely continuous norm on the unit on a finite atomless measure space (cf. Proposition* 2.3*);*
- *every representable and thus, every weakly compact operator $T \in \mathcal{L}(L_1(\mu), X)$, for any Banach space X (cf. Proposition* 2.4*);*
- *every ℓ_1-strictly singular operator $T \in \mathcal{L}(L_1, X)$, for any Banach space X (cf. Theorem* 7.2*);*
- *every operator $T \in \mathcal{L}(E, c_0(\Gamma))$, where E is a Köthe F-space over the reals on a finite atomless measure space (Ω, Σ, μ) for which there exists a reflexive Köthe–Banach space E_1 on (Ω, Σ, μ) with continuous inclusion embedding $E_1 \subseteq E$ (cf. Theorem* 9.3*);*
- *every operator $T \in \mathcal{L}(L_p, L_r)$, for $1 \leq p < 2$ and $p < r < \infty$ (cf. Theorem* 9.7*);*
- *every operator $T \in \mathcal{L}(L_p, \ell_r)$, where $1 \leq p, r < \infty$, and either $r \neq 2$ or $r = 2$ and $p < 2$ (cf. Theorem* 9.9*);*
- *every ℓ_2-strictly singular operator $T : L_p \to X$, for p with $1 < p < \infty$, and any Banach space X with an unconditional basis (cf. Theorem* 9.15*);*
- *every non-Enflo operator $T \in \mathcal{L}(L_p)$, for $1 \leq p < 2$ (cf. Theorems* 7.45 *and* 7.55*).*

In fact, it follows from Theorem 7.80 that operators from the last item in Corollary 11.4 are the only hereditarily narrow operators on L_1. That is, we have the following characterization.

Theorem 11.5. *An operator $T \in \mathcal{L}(L_1)$ is hereditarily narrow if and only if T is non-Enflo.*

As a consequence, by Theorem 10.41, we obtain that the set of all non-Enflo operators on L_1 is a band in $\mathcal{L}(L_1)$ (this was observed by Liu in [81]).

Note that Theorem 11.5 does not hold for L_p with $2 < p < \infty$ (see Example 7.56). We do not know whether it holds for $1 < p < 2$.

Open problem 11.6. Let $1 < p < 2$. Is every hereditarily narrow operator $T \in \mathcal{L}(L_p)$ non-Enflo?

We remark that an affirmative answer to Open problem 7.81 for $1 < p < 2$, would imply the same answer to Open problem 11.6.

Next we show that the class of hereditarily narrow operators is strictly larger than the classes of AM-compact and Dunford–Pettis operators. To do this, we use the example from Proposition 4.7 for $E = L_p(\mu)$.

Proposition 11.7. *Let* (Ω, Σ, μ) *be a finite atomless measure space,* $1 \le p < \infty$, $p \neq 2$ *and let* $\mathcal{G}$ *be a purely atomic sub-*σ*-algebra of* Σ *with the atoms* $(A_i)_{i\in I}$*. Then the conditional expectation operator*

$$M^{\mathcal{G}} x = \sum_{i\in I} \Big(\frac{1}{\mu(A_i)} \int_{A_i} x \, \mathrm{d}\mu \Big) \cdot \mathbf{1}_{A_i}$$

is a hereditarily narrow operator on $L_p(\mu)$ *which is not Dunford–Pettis and non-AM-compact.*

Proof. Since $M^{\mathcal{G}}$ fixes a copy of ℓ_p, it is not Dunford–Pettis and non-AM-compact. The fact that $M^{\mathcal{G}}$ is hereditarily narrow follows from

- Rosenthal's Theorem 7.45, for $p = 1$;
- Johnson–Maurey–Schechtman–Tzafriri's Theorem 7.55, for $1 < p < 2$;
- Mykhaylyuk–Popov–Randrianantoanina–Schechtman's Theorem 9.9, for $2 < p < \infty$. □

Another interesting example is the natural projection of L_p onto the subspace spanned by the Rademacher system, which is hereditarily narrow when $1 < p < 2$.

Proposition 11.8. *Let* $1 < p < 2$*. Then the orthogonal projection*

$$Px = \sum_{n=1}^{\infty} \Big(\int_{[0,1]} x \cdot r_n \, \mathrm{d}\mu \Big) r_n \tag{11.1}$$

of L_p *onto the span* R *of the Rademacher system* (r_n) *is hereditarily narrow, but it is neither AM-compact, nor Dunford–Pettis.*

Proof. The boundedness of P is explained in [79, p. 72]. Hereditary narrowness follows from Theorem 7.55. Since P is a projection onto R, it is neither AM-compact, nor Dunford–Pettis. □

It is well known that the subspace R is uncomplemented in L_1, as well as in any reflexive subspace of L_1. However, Rosenthal proved in [123] that there exists $\delta > 0$ such that for every $\varepsilon \in (0, \delta)$ there exists $T_\varepsilon \in \mathcal{L}(L_1)$ such that $T w_I = \varepsilon^{|I|} w_I$ for each finite set I of the integers $\mathbb{N}$, where (w_I) is the Walsh system (cf. Definition 1.4). The operator T_ε is called the ε-biased coin convolution operator. Among the properties of T_ε we mention that T_ε restricted to R is the identity multiplied by ε. Hence,

T_ε is not Dunford–Pettis, not AM-compact and not representable, and it is non-Enflo, and hence, it is hereditarily narrow.

We do not know whether Proposition 11.8 is true for $2 < p < \infty$.

Open problem 11.9. Let $2 < p < \infty$. Is the orthogonal projection P, defined by (11.1), from L_p onto the span R of the Rademacher system (r_n), hereditarily narrow?

It is not hard to show that if the projection P defined by (11.1) is well defined and bounded on a r.i. space E on $[0,1]$ then it is narrow. Indeed, if a set $B \in \Sigma$ is a union of dyadic intervals $I_n^k = [(k-1)/2^n, k/2^n)$ of a fixed level n, then for $y = r_{n+1} \cdot r_{n+2} \cdot \mathbf{1}_B$ we have $Py = 0$. Given any $A \in \Sigma$ and $\varepsilon > 0$, we choose $n \in \mathbb{N}$ and a set $B \in \Sigma$ which is a union of dyadic intervals $I_n^k = [(k-1)/2^n, k/2^n)$ of level n so that $\mu(A \triangle B) < \varepsilon/\|P\|$. Then $x = r_{n+1} \cdot r_{n+2} \cdot \mathbf{1}_A$ is a sign on A, and by the above argument, for $y = r_{n+1} \cdot r_{n+2} \cdot \mathbf{1}_B$ we obtain $\|Px\| \le \|P\|\|x - y\| < \varepsilon$.

More generally, we can prove that every operator that sends the Walsh system to a norm null system as the number of the Rademacher factors grows to infinity then the operator is narrow.

Proposition 11.10. *Let E be a Köthe–Banach space on* $[0,1]$ *with an absolutely continuous norm on the unit, and X be a Banach space. If $T \in \mathcal{L}(E, X)$ has the property*

$$\lim_{|I| \to \infty} T w_I = 0$$

then T is narrow.

Proof. By Corollary 2.11, the Walsh system (w_I), whose linear span is equal to the span of all simple functions, is complete in E. Fix any $A \in \Sigma^+$ and $\varepsilon > 0$. We approximate $\mathbf{1}_A$ in E by a suitable linear combination of (w_I) so that

$$\Big\| \mathbf{1}_A - \sum_{k=1}^{m} a_k w_{I_k} \Big\| < \frac{\varepsilon}{2\|T\|} . \tag{11.2}$$

Choose $n_0 \in \mathbb{N}$ so that if $I \in \mathbb{N}^{<\omega}$ with $|I| \ge n_0$ then $\|T w_I\| < \varepsilon/(2 \max_k |a_k|)$. Since $I_0 = \bigcup_{k=1}^{n} I_k$ is finite, we can choose $J \in \mathbb{N}^{<\omega}$ with $J \subset \mathbb{N} \setminus I_0$ and $|J| = n_0$. Let $x = w_J \cdot \mathbf{1}_A$. Obviously, x is a sign on A. We show that $\|Tx\| < \varepsilon$. Observe that

$$\Big| x - \sum_{k=1}^{m} a_k w_{I_k \cup J} \Big| = \Big| w_J \Big(\mathbf{1}_A - \sum_{k=1}^{m} a_k w_{I_k} \Big) \Big| = \Big| \mathbf{1}_A - \sum_{k=1}^{m} a_k w_{I_k} \Big|$$

and hence by (11.2), $\|x - y\| < \varepsilon/(2\|T\|)$ where $y = \sum_{k=1}^{m} a_k w_{I_k \cup J}$. Then

$$\|Tx\| \le \|Ty\| + \|T\|\|x - y\| < \|Ty\| + \frac{\varepsilon}{2} .$$

It remains to show that $\|Ty\| < \varepsilon/2$:

$$\|Ty\| \le \sum_{k=1}^{m} |a_k| \left| T w_{I_k \cup J} \right| < \sum_{k=1}^{m} |a_k| \frac{\varepsilon}{2m|a_k|} = \frac{\varepsilon}{2} .$$

By Proposition 1.9, T is narrow (the sign x we have constructed is not necessarily of mean zero). □

Kadets–Kalton–Werner's theorem

The aim of the rest of the section is to prove the following nice theorem of V. Kadets, Kalton and Werner [53].

Theorem 11.11. *Let X be a Banach space with an unconditional basis. Then every operator $T \in \mathcal{L}(L_1, X)$ is hereditarily narrow.*

An important consequence is the following Rosenthal's unpublished generalization of the Pełczyński theorem that L_1 cannot be isomorphically embedded in a Banach space with an unconditional basis [105].

Corollary 11.12. *The space L_1 does not sign-embed in a Banach space with an unconditional basis.*

The key idea of the proof is the following result which is of independent interest.

Theorem 11.13. *Let X be a Banach space. Then the pointwise unconditional sum $T = \sum_{n=1}^{\infty} T_n$ of hereditarily narrow operators $T_n \in \mathcal{L}(L_1, X)$ is hereditarily narrow.*

First, we show that Theorem 11.11 is an immediate consequence of Theorem 11.13.

Proof of Theorem 11.11. Let (P_n) be the basis projections associated with an unconditional basis of X and $T \in \mathcal{L}(L_1, X)$ be any operator. Then $T = \sum_{n=1}^{\infty}(P_{n+1} - P_n)T$ is a pointwise unconditionally convergent series of rank-one operators, which are hereditarily narrow. By Theorem 11.13, T is hereditarily narrow. □

For the proof of Theorem 11.13, we need several lemmas. To formulate the first of them, we observe that, if $T = \sum_{n=1}^{\infty} T_n$ is a pointwise unconditionally convergent series of operators $T_n : E \to X$ then, by the Banach–Steinhaus theorem, the number

$$M = \sup_{\theta_n = \pm 1} \left\| \sum_{n=1}^{\infty} \theta_n T_n \right\| \tag{11.3}$$

is finite.

Lemma 11.14. *Let $1 \le p < \infty$, X be a Banach space, and $T_n : L_p \to X$ be hereditarily narrow operators with pointwise unconditionally convergent series $T = \sum_{n=1}^{\infty} T_n$. Then, for any $\varepsilon \in (0, 1/2)$, there exists a Banach space Y, operators $W \in \mathcal{L}(Y, X)$ and $\widetilde{T} \in \mathcal{L}(L_p, Y)$ with $\|W\| \le 1$ and $T = W \circ \widetilde{T}$, $\|\widetilde{T}\| \le M$, where M is the constant defined by* (11.3)*, and there exists an atomless sub-σ-algebra $\Sigma_1 \subseteq \Sigma$, a system $(g_n)_{n=1}^{\infty}$ which is a basis in $L_p(\Sigma_1)$ isometrically equivalent to the L_∞-normalized Haar system, and operators $U, V \in \mathcal{L}(L_p(\Sigma_1), Y)$ with $U + V = \widetilde{T}|_{L_p(\Sigma_1)}$ such that $\|V\| \le \varepsilon$ and $(Ug_n)_{n=1}^{\infty}$ is a 1-unconditional system.*

Proof. We define Y to be the linear space of all sequences $y = (y_1, y_2, \ldots)$, $y_i \in X$, such that the series $\sum_{n=1}^{\infty} y_n$ converges unconditionally, equipped with the norm

$$\|y\| = \sup_{\theta_n = \pm 1} \left\| \sum_{n=1}^{\infty} \theta_n y_n \right\| .$$

For each $x \in L_p$ and for each $y = (y_1, y_2, \ldots) \in Y$ we set $\widetilde{T}x = (T_1 x, T_2 x, \ldots) \in Y$ and $Wy = \sum_{n=1}^{\infty} y_n \in X$. Obviously, $T = W \circ \widetilde{T}$ and $\|W\| \le 1$.

Given $n, m \in \mathbb{N}$ with $n < m$, we define contractive projections $P_{n,m}$ by setting $P_{n,m}\, y = (0, \ldots, 0, y_n, y_{n+1}, \ldots, y_{m-1}, 0, 0, \ldots)$ for each $y = (y_1, y_2, \ldots) \in Y$, and $P_{n,\infty}$ by $P_{n,\infty} y = (0, \ldots, 0, y_n, y_{n+1}, \ldots)$. Observe that $\lim_{n\to\infty} P_{n,\infty}\, y = 0$ for each $y \in Y$.

Fix any $\varepsilon > 0$. For the first step, we set $g_1 = \mathbf{1}_{[0,1]}$, and choose $n_1 \in \mathbb{N}$ so that $\|P_{n_1} \widetilde{T} g_1\| < \varepsilon/2$. We put $Ug_1 = P_{1,n_1} \widetilde{T} g_1$ and $Vg_1 = P_{n_1,\infty} \widetilde{T} g_1$. Denote $A_{1,1} = \{t \in [0,1] : g_1(t) = 1\}$ and $A_{1,2} = \{t \in [0,1] : g_1(t) = -1\}$. Observe that, by Proposition 11.2, the operator $P_{1,n_1}\widetilde{T}$ as a finite sum of hereditarily narrow operator is hereditarily narrow, and hence, narrow. We choose a mean zero sign g_2 on $A_{1,1}$ such that $\|P_{1,n_1} \widetilde{T} g_2\| \le \varepsilon \|g_2\|/8$, and find $n_2 > n_1$ such that $\|P_{n_2,\infty} \widetilde{T} g_2\| \le \varepsilon\|g_2\|/8$. We set $Ug_2 = P_{n_1,n_2} \widetilde{T} g_2$ and $Vg_2 = (P_{1,n_1} + P_{n_2,\infty})\widetilde{T} g_2$. Note that $\|Vg_2\| \le 2^{-2} \varepsilon \|g_2\|$.

Continuing the procedure in this fashion, we obtain a sequence $(g_n)_{n=1}^{\infty}$ isometrically equivalent to the L_∞-normalized Haar system in L_p, and operators U, V: $[g_n] \to Y$ such that $U + V = \widetilde{T}|_{L_p(\Sigma_1)}$ and

$$\|Vg_n\| \le 2^{-n} \varepsilon \|g_n\|, \quad \text{for each } n \in \mathbb{N} , \tag{11.4}$$

where Σ_1 is the sub-σ-algebra of Σ generated by $(g_n)_{n=1}^{\infty}$. The boundedness of V and the inequality $\|V\| \le \varepsilon$ follow from (11.4), and the boundedness of U follows from the equality $Ug_n + Vg_n = \widetilde{T} g_n$ for each $n \in \mathbb{N}$. It remains to observe that $(Ug_n)_{n=1}^{\infty}$ is a disjoint sequence, and hence, is 1-unconditional by the definition of the norm in Y. □

Lemma 11.15. *Let $(\overline{h}_j)_{j=1}^{\infty}$ be the L_∞-normalized Haar system, X a Banach space, and $U \in \mathcal{L}(L_1, X)$ an operator such that $(U\overline{h}_j)_{j=1}^{\infty}$ is 1-unconditional. Then for each $x = \sum_{j=1}^{\infty} a_j \overline{h}_j \in L_1$, we have $\|Ux\| \le 2\|U\| \sup_j |a_j|$.*

Proof. Fix any $n \in \mathbb{N}$ and $m = 1, \dots, 2^n$, and let $y_{n,m} = 2^n \mathbf{1}_{I_n^m} - \mathbf{1}$. Then

$$\|y_{n,m}\| \le 2^n \|\mathbf{1}_{I_n^m}\| + \|\mathbf{1}\| \le 2\,. \tag{11.5}$$

Since $\int_{[0,1]} y_{n,m} \cdot \overline{h}_1 \, \mathrm{d}\mu = 0$ and $\int_{[0,1]} y_{n,m} \cdot \overline{h}_{2^i+k} \, \mathrm{d}\mu = 0$ for each $i \ge n$ and $k = 1, \dots, 2^i$, we can write

$$y_{n,m} = \sum_{i=0}^{n-1} \sum_{k=1}^{2^i} \Big(\int_{[0,1]} y_{n,m} \cdot h_{2^i+k} \, \mathrm{d}\mu \Big) \overline{h}_{2^i+k}\,, \tag{11.6}$$

where (h_j) is the L_1-normalized Haar system. Observe that

$$\int_{[0,1]} y_{n,m} \cdot h_{2^i+k} \, \mathrm{d}\mu = \int_{[0,1]} 2^n \mathbf{1}_{I_n^m} \cdot h_{2^i+k} \, \mathrm{d}\mu = 2^n \int_{I_n^m} h_{2^i+k} \, \mathrm{d}\mu$$
$$= \begin{cases} \theta_{i,m} \cdot 2^i, & \text{if } I_n^m \subset \operatorname{supp} h_{2^i+k} \\ 0, & \text{else} \end{cases},$$

where $\theta_{i,m} \in \{-1, 1\}$. Hence we can continue (11.6) as follows:

$$y_{n,m} = \sum_{i=0}^{n-1} \theta_{i,m} \cdot 2^i \cdot \overline{h}_{2^i+k(i,m)}\,,$$

where $k(i,m)$ is the unique number such that $I_n^m \subset \operatorname{supp} h_{2^i+k(i,m)}$. By (11.5) and the 1-unconditionality of $(U\overline{h}_j)_{j=1}^{\infty}$, we get

$$\Big\| \sum_{i=0}^{n-1} 2^i U \overline{h}_{2^i+k(i,m)} \Big\| \le 2\|U\|\,.$$

Averaging the last inequality over all $m = 1, \dots 2^n$, we obtain

$$2\|U\| \ge \Big\| \frac{1}{2^n} \sum_{m=1}^{2^n} \Big(U\overline{h}_{1+k(1,m)} + 2U\overline{h}_{2+k(2,m)} + \dots + 2^{n-1} U\overline{h}_{2^{n-1}+k(n-1,m)} \Big) \Big\|\,.$$

Since for any fixed i and k, the term $2^i U\overline{h}_{2^i+k}$ occurs 2^{n-i} times, we have

$$2\|U\| \ge \Big\| \sum_{i=0}^{n-1} \sum_{k=1}^{2^i} U\overline{h}_{2^i+k} \Big\|\,.$$

By the 1-unconditionality of $(U\overline{h}_j)$, this implies that for each $x = \sum_{j=1}^{\infty} a_j \overline{h}_j \in L_1$

$$\Big\| \sum_{j=1}^{2^{n-1}} a_j U\overline{h}_j \Big\| = \Big\| \sum_{i=0}^{n-1} \sum_{k=1}^{2^i} a_{2^i+k} U\overline{h}_{2^i+k} \Big\|$$
$$\le \sup_j |a_j| \Big\| \sum_{i=0}^{n-1} \sum_{k=1}^{2^i} U\overline{h}_{2^i+k} \Big\| \le 2\|U\| \sup_j |a_j|.$$

By continuity of the norm, $\|Ux\| \le 2\|U\| \sup_j |a_j|$. □

In the next lemma, it is convenient to enumerate the Haar system in a different way.

Lemma 11.16. *Let* $(\overline{h}_{0,0}) \cup (\overline{h}_{n,k})_{n=0}^{\infty}{}_{k=1}^{2^n}$ *be the* L_∞*-normalized Haar system. Then for every* $\varepsilon > 0$*, there exists a sign* x *on* $[0,1]$ *of the form* $x = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \alpha_{n,k} \overline{h}_{n,k}$ *such that* $|\alpha_{n,k}| \le \varepsilon$ *for each* $n = 0, 1, \ldots$ *and* $k = 1, \ldots, 2^n$.

Proof. Let $m \in \mathbb{N}$ so that $m^{-1} \le \varepsilon$. We define recursively a sequence $(x_n)_{n=0}^{\infty}$ by setting $x_0 = m^{-1}\overline{h}_{0,1}$ and $x_n = \sum_{k=1}^{2^n} \alpha_{n,k} \overline{h}_{n,k}$, for $n \ge 1$, where

$$\alpha_{n,k} = \begin{cases} \frac{1}{m}, & \text{if } \left|\sum_{i=1}^{n-1} x_i\right| < 1 \text{ on } \operatorname{supp} \overline{h}_{n,k}, \\ 0, & \text{else.} \end{cases}$$

By the construction, $|\sum_{i=1}^{n-1} x_i| \le 1$ on $[0,1]$ for each $n \ge 1$, since x_k takes values from $\{0, m^{-1}\}$ only. Thus, all partial sums of the series $x = \sum_{n=0}^{\infty} x_n$ are bounded in modulus by 1. Since (x_n) is an orthogonal system in L_2, the series converges in L_2, and hence, in L_1. By the construction, $x = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \alpha_{n,k} \overline{h}_{n,k}$, and $|\alpha_{n,k}| \le \varepsilon$ for each $n = 0, 1, \ldots$ and $k = 1, \ldots, 2^n$. It remains to show that x is a sign on $[0,1]$. We set $A = \{t \in [0,1] : |x(t)| \ne 1\}$. By the construction, for each $n = 0, 1, \ldots$

$$A \subseteq \big\{t \in [0,1] : x_n(t) \ne 0\big\} = \Big\{t \in [0,1] : |x(t)| = \frac{1}{m}\Big\}.$$

Thus $\mu(A) \le m\|x_n\|$ for each n. Since $\lim_{n\to\infty} \|x_n\| = 0$, we get $\mu(A) = 0$. □

We note that the idea behind the proof of Lemma 11.16 is the stopping time of a martingale, similar to the idea of the proof of Step 1 of case (iii) of Theorem 9.9. The idea of applying a stopping time in this way originated in [59]. Below we outline the proof of Lemma 11.16 using martingale terminology.

For simplicity of notation we will work with the classical Haar system $h_1, h_2, \ldots$ on $[0,1]$. Let $\xi_n = \sum_{k=1}^{n} h_k$ and $T = \inf\{n : |\xi_n| \ge m\}$. Then (ξ_n) is a martingale, T is a stopping time and $(\xi_n') = (\xi_{n \wedge T})$ is a uniformly bounded martingale. Hence (ξ_n') converges almost surely and in L_1 to a limit ξ that takes only the values $\pm m$ on $\{T < \infty\}$, but since (ξ_n) fails to converge pointwise, the event $\{T = \infty\}$ has probability 0. This shows that $\xi = \pm m$ almost surely and $\mathbb{E}\xi = 0$. Hence $f = \xi/m$ is the sign that we are seeking.

Proof of Theorem 11.13. Fix any $A \in \Sigma^+$, any atomless sub-σ-algebra $\mathcal{F}$ of $\Sigma(A)$, any $B \in \mathcal{F}$ and any $\varepsilon > 0$. Applying Lemma 11.14 to the restrictions of T_n and T to $L_1(\mathcal{F}(B))$, we obtain a system (g_n) isometrically equivalent to the L_∞-normalized Haar system in L_1, a Banach space Y, and operators $U, V \in \mathcal{L}(L_1(\mathcal{F}(B)), Y)$, $W \in \mathcal{L}(Y, X)$ such that $\|W\| \le 1$, $T = W \circ (U + V)$ on $L_1(\mathcal{F}(B))$, $\|V\| \le \varepsilon/2$ and

(Ug_n) is 1-unconditional. By Lemma 11.16 there exists a sign $x = \sum_{j=2}^{\infty} \alpha_j g_j$ on B such that $|\alpha_j| \le \varepsilon/(4\|U\|)$ for each $j \ge 2$. By Lemma 11.15 we obtain that

$$\|Ux\| \le 2\|U\| \frac{\varepsilon}{4\|U\|} = \frac{\varepsilon}{2} .$$

Therefore, $\|Tx\| \le \|Ux\| + \|Vx\| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

11.2 Gentle narrow operators on L_p with $1 < p \le 2$

In this section we study Open problem 7.52. This problem essentially asks whether in the definition of a narrow operator one could drop the restrictions on the form of the element x with $\|Tx\| < \varepsilon$. In Definition 1.5, x is required to be $\{1, -1, 0\}$-valued, while in Theorem 7.30, x is of any form with $\|x\| = 1$. In this section we identify a condition, called p-gentle, on the rate of growth of the distribution of x and we say that an operator T is gentle narrow if for every $\varepsilon > 0$ and every measurable subset $A \subseteq [0, 1]$, there exists an x supported on A and with a p-gentle the rate of growth so that $\|Tx\| < \varepsilon$ (see Definition 11.19). We prove that on L_p, $1 < p < 2$, such operators have to be narrow, which partially answers Open problem 7.52.

Results of this section were obtained in [102].

Definition 11.17. For any $x \in L_0$ and $M > 0$ we define the M-truncation x^M of x by setting

$$x^M(t) = \begin{cases} x(t), & \text{if } \bigl|x(t)\bigr| \le M, \\ M \cdot \operatorname{sign}\bigl(x(t)\bigr), & \text{if } \bigl|x(t)\bigr| > M. \end{cases}$$

Definition 11.18. Let $1 < p \le 2$. A decreasing function $\varphi : (0, +\infty) \to [0, 1]$ is called *p-gentle* if

$$\lim_{M \to +\infty} M^{2-p} \bigl(\varphi(M)\bigr)^p = 0 .$$

Definition 11.19. Let $1 < p \le 2$, and let X be a Banach space. We say that an operator $T \in \mathcal{L}(L_p, X)$ is *gentle narrow* if there exists a p-gentle function $\varphi : (0, +\infty) \to [0, 1]$ such that for every $\varepsilon > 0$, every $M > 0$ and every $A \in \Sigma$ there exists $x \in L_p(A)$ such that the following conditions hold:

(i) $\|x\| = \mu(A)^{1/p}$;

(ii) $\|x - x^M\| \le \varphi(M)\, \mu(A)^{1/p}$;

(iii) $\|Tx\| \le \varepsilon$.

Observe that every narrow operator is gentle narrow with

$$\varphi(M) = \begin{cases} 1 - M, & \text{if } 0 \le M < 1, \\ 0, & \text{if } M \ge 1. \end{cases}$$

Indeed, for every sign x on A, $\|x - x^M\| = \varphi(M)\mu(A)^{1/p}$ for each $M \geq 0$, where φ is the function defined above.

Another condition of an operator $T \in \mathcal{L}(L_p, X)$ implying that T is gentle narrow is that for each $A \in \Sigma$ and each $\varepsilon > 0$ there exists a mean zero Gaussian random variable $x \in L_p(A)$ with the distribution

$$d_x \overset{\text{def}}{=} \mu\{t \in [0,1] : x(t) < a\} = \frac{\mu(A)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{a} e^{-\frac{t^2}{2\sigma^2}} dt$$

and such that $\|Tx\| < \varepsilon$. One can show that in this case T is gentle narrow with $\varphi(M) = Ce^{-\frac{M^2}{2\sigma^2}}$, where C is a constant independent of M.

The main result of this section is the following theorem.

Theorem 11.20. *Let $1 < p \leq 2$. Then every gentle narrow operator $T \in \mathcal{L}(L_p)$ is narrow.*

For the proof, we need several lemmas. First of them asserts that in the definition of a gentle narrow operator, in addition to (i)–(iii) we can claim one more property.

Lemma 11.21. *Suppose $1 < p \leq 2$, X is a Banach space and $T \in \mathcal{L}(L_p, X)$ is a gentle narrow operator with a gentle function $\varphi : [0, +\infty) \to [0, 1]$. Then for every $\varepsilon > 0$, every $M > 0$ and every $A \in \Sigma$ there exists $x \in L_p(A)$ such that the following conditions hold*

(i) $\|x\| = \mu(A)^{1/p}$;

(ii) $\|x - x^M\| \leq \varphi(M)\, \mu(A)^{1/p}$;

(iii) $\|Tx\| \leq \varepsilon$;

(iv) $\int_{[0,1]} x \, \mathrm{d}\mu = 0$.

Proof of Lemma 11.21. Without loss of generality we assume that $\|T\| = 1$. Fix $\varepsilon > 0$, $M > 0$ and $A \in \Sigma$. Let $n \in \mathbb{N}$ so that $(\mu(A)/n)^{1/p} < \varepsilon/4$, and decompose $A = A_1 \sqcup \ldots \sqcup A_n$ with $A_k \in \Sigma$ and $\mu(A_k) = \mu(A)/n$ for every $k = 1, \ldots, n$. By the definition of a gentle narrow operator, for each $k = 1, \ldots, n$, there exist $x_k \in L_p(A_k)$ so that $\|x_k\|^p = \mu(A)/n$, $\|x_k - x_k^M\| \leq \varphi(M)\mu(A)^{1/p}$, and $\|Tx_k\| < \varepsilon/(2n)$.

Without loss of generality, we may and do assume that

$$\delta = \left| \int_{[0,1]} x_n \, \mathrm{d}\mu \right| \geq \left| \int_{[0,1]} x_k \, \mathrm{d}\mu \right|,$$

for $k = 1, \ldots, n-1$ (otherwise we rearrange $A_1, \ldots, A_n$). Observe that

$$\|x_k\| = \left(\frac{\mu(A)}{n}\right)^{1/p} < \frac{\varepsilon}{4}. \tag{11.7}$$

Inductively, we choose sign numbers $\theta_1 = 1$ and $\theta_2, \dots, \theta_{n-1} \in \{-1, 1\}$ so that for each $k = 1, \dots, n-1$, we have

$$\Big|\int_{[0,1]} \sum_{i=1}^{k} \theta_i x_i \, d\mu\Big| \le \delta .$$

Choose a sign r on A_n so that

$$\int_{[0,1]} r x_n \, d\mu = -\int_{[0,1]} \sum_{i=1}^{n-1} \theta_i x_i \, d\mu . \tag{11.8}$$

This is possible, because

$$\Big|\int_{[0,1]} \sum_{i=1}^{n-1} \theta_i x_i \, d\mu\Big| \le \delta = \Big|\int_{[0,1]} x_n \, d\mu\Big|.$$

We set $x = \sum_{i=1}^{n-1} \theta_k x_k + r x_n$ and show that x satisfies the desired properties:
(i) $\|x\|^p = \sum_{k=1}^{n} \|x_k\|^p = n \cdot \frac{\mu(A)}{n} = \mu(A)$;
(ii) $\|x - x^M\|^p = \sum_{k=1}^{n} \|x_k - x_k^M\|^p \le n \cdot (\varphi(M))^p \cdot \frac{\mu(A)}{n} = (\varphi(M))^p \mu(A)$;
(iii) Since $\|T\| = 1$, by (11.7) and the definition of x we get

$$\|Tx\| \le \sum_{k=1}^{n-1} \|Tx_k\| + \|Trx_n\| \le \sum_{k=1}^{n} \|Tx_k\| + \|x_n - rx_n\| < \frac{\varepsilon}{2} + \|2x_n\| < \varepsilon .$$

Property (iv) for x follows from (11.8). □

Lemma 11.22. *Assume* $1 < p \le 2$, $a > 0$ *and* $|b| \le a$. *Then*

$$(a + b)^p - p\, a^{p-1} b \ge a^p + \frac{p\,(p-1)}{2^{3-p}} \cdot \frac{b^2}{a^{2-p}} . \tag{11.9}$$

Proof of Lemma 11.22. Dividing the inequality by a^p and denoting $t = b/a$, we pass to an equivalent inequality

$$f(t) \overset{\text{def}}{=} (1+t)^p - pt - 1 - \frac{p(p-1)}{2^{3-p}} t^2 \ge 0$$

for each $t \in [-1, 1]$, which we have to prove. Observe that

$$f'(t) = p(1+t)^{p-1} - p - \frac{p(p-1)}{2^{2-p}} t \quad \text{and} \quad f''(t) = \frac{p(p-1)}{(1+t)^{2-p}} - \frac{p(p-1)}{2^{2-p}} .$$

Since $f''(t) > 0$ for every $t \in (-1, 1)$ and $f'(0) = 0$, $t_0 = 0$ is the point of a global minimum of $f(t)$ on $[-1, 1]$. Since $f(0) = 0$, the inequality (11.9) follows. □

Lemma 11.23. *Assume* $1 < p \le 2$, $A \in \Sigma$, $a \ne 0$, $y \in L_\infty(A)$, $|y(t)| \le |a|$ *for each* $t \in A$ *and* $\int_{[0,1]} y \, d\mu = 0$. *Then*

$$\|a\mathbf{1}_A + y\|_p^p \ge |a|^p \mu(A) + \frac{p\,(p-1)}{2^{3-p}} \cdot \frac{\|y\|_2^2}{|a|^{2-p}} \, .$$

Proof of Lemma 11.23. Evidently, it is enough to consider the case when $a > 0$ which one can prove by integrating inequality (11.9) written for $b = y(t)$. □

Lemma 11.24. *Let* $1 < p \le 2$ *and* $T \in \mathcal{L}(L_p)$ *be a gentle narrow operator with a gentle function* $\varphi : [0, +\infty) \to [0, 1]$ *and* $\|T\| = 1$. *Then for every* $M > 0$, *every* $\delta > 0$, *every* $B \in \Sigma^+$, *every* $\eta \in (0, 1/2)$, *and every* $y \in L_p$ *with* $\eta \le |y(t)| \le 1 - \eta$ *for all* $t \in B$, *there exists* $h \in B_{L_\infty(B)}$ *satisfying the following properties:*

(a) $\|Th\| < 2\,\mu(B)^{1/p} \dfrac{\varphi(M)}{M}$;

(b) $\|y \pm \eta h\|^p > \|y\|^p + \dfrac{p\,(p-1)}{2^{3-p}} \cdot \dfrac{\eta^2}{(1-\eta)^{2-p}} \cdot \mu(B) \cdot \dfrac{(1-\varphi(M))^2}{M^2} - \delta.$

Proof of Lemma 11.24. Fix M, δ, B, η and y as in the assumptions of the lemma. Since we need to prove strict inequalities, we may and do assume without loss of generality that y is a simple function on B

$$y \cdot \mathbf{1}_B = \sum_{k=1}^m b_k \mathbf{1}_{B_k}, \quad B = B_1 \sqcup \dots \sqcup B_m, \quad \eta \le |b_k| \le 1 - \eta \, . \tag{11.10}$$

For each $k = 1, \dots, m$ we choose $x_k \in L_p(B_k)$ so that

(i) $\|x_k\|^p = \mu(B_k)$;

(ii) $\|x_k - x_k^M\| \le \varphi(M)\mu(B_k)^{1/p}$;

(iii) $\|Tx_k\| \le \dfrac{\varphi(M)\mu(B)^{1/p}}{m}$;

(iv) $\displaystyle\int_{[0,1]} x_k \, d\mu = 0.$

Let $x = \sum_{k=1}^m x_k$ and $h = M^{-1}x^M$. We show that h has the desired properties:
(a) Observe that (iii) implies that

$$\|Tx\| \le \sum_{k=1}^m \|Tx_k\| < m \frac{\varphi(M)\,\mu(B)^{1/p}}{m} = \varphi(M)\,\mu(B)^{1/p} \, , \tag{11.11}$$

and (ii) yields

$$\|x - x^M\|^p = \sum_{k=1}^m \|x_k - x_k^M\|^p \le \sum_{k=1}^m \big(\varphi(M)\big)^p \mu(B_k) = \big(\varphi(M)\big)^p \mu(B). \tag{11.12}$$

Combining (11.11) and (11.12), we get

$$\|Th\| = \frac{\|T(x^M)\|}{M} \le \frac{\|Tx\|}{M} + \frac{\|x - x^M\|}{M} < 2\,\mu(B)^{1/p}\frac{\varphi(M)}{M}\,.$$

(b) Using the well-known inequality for norms in L_p and L_2 (see [25, p. 73]), (i) and (11.12) we obtain

$$\begin{aligned}\|x^M\|_2 &\ge \|x^M\|_p\,\mu(B)^{1/2-1/p} \ge \big(\|x\| - \|x - x^M\|\big)\mu(B)^{1/2-1/p}\\ &\ge \mu(B)^{1/p}\big(1-\varphi(M)\big)\,\mu(B)^{1/2-1/p} = \big(1-\varphi(M)\big)\mu(B)^{1/2}\,,\end{aligned}$$

and hence,

$$\|x^M\|_2^2 \ge \big(1-\varphi(M)\big)^2\,\mu(B)\,. \tag{11.13}$$

Thus,

$$\|y \pm \eta h\|^p = \big\|y \cdot \mathbf{1}_{[0,1]\setminus B}\big\|^p + \sum_{k=1}^{m}\big\|b_k \cdot \mathbf{1}_{B_k} \pm \eta M^{-1}x_k^M\big\|$$

$$\overset{\text{by Lemma 11.23}}{\ge} \big\|y \cdot \mathbf{1}_{[0,1]\setminus B}\big\|^p + \sum_{k=1}^{m}|b_k|^p\mu(B_k) + \frac{p\,(p-1)}{2^{3-p}}\cdot\frac{\eta^2}{M^2}\sum_{k=1}^{m}\frac{\|x_k^M\|_2^2}{|b_k|^{2-p}}$$

$\Big($using (11.13), $|b_k| \le 1-\eta$ and the equality $\sum_{k=1}^m \|x_k^M\|_2^2 = \|x^M\|_2^2\Big)$

$$\ge \|y\|^p + \frac{p\,(p-1)}{2^{3-p}}\cdot\frac{\eta^2}{(1-\eta)^{2-p}}\cdot\mu(B)\cdot\frac{\big(1-\varphi(M)\big)^2}{M^2}\,.$$

□

Note that the reason for including δ in the second inequality is to make possible the reduction to simple functions in the proof.

Proof of Theorem 11.20. Let $T \in \mathcal{L}(L_p)$ be a gentle narrow operator with a p-gentle function $\varphi : [0, +\infty) \to [0, 1]$. To prove that T is narrow, by Theorem 7.59, it is enough to prove that it is somewhat narrow. Without loss of generality, we may and do assume that $\|T\| = 1$. Fix any $A \in \Sigma^+$ and $\varepsilon > 0$, and prove that there exists a sign $x \in L_p(A)$ such that $\|Tx\| < \varepsilon\|x\|$. Consider the set

$$K_\varepsilon = \{y \in B_{L_\infty(A)} : \|Ty\| \le \varepsilon\|y\|\}\,.$$

By arbitrariness of ε, it is enough to prove the following statement:

$$\big(\forall\varepsilon_1 > 0\big)\big(\exists \text{ a sign } x \in L_p(A)\big)\big(\exists y \in K_\varepsilon\big) : \ \|x - y\| < \varepsilon_1\|y\|\,. \tag{11.14}$$

Indeed, if (11.14) is true for each ε and ε_1, we choose a sign $x \in L_p(A)$ and $y \in K_{\varepsilon/2}$ such that $\|x - y\| < \varepsilon_1\|y\|$ where

$$\varepsilon_1 = \frac{\varepsilon}{2\varepsilon + 2}\,. \tag{11.15}$$

Since $\varepsilon_1 < 1$, the inequality $\|y\| \le \|x\| + \|x - y\| < \|x\| + \varepsilon_1 \|y\|$ implies

$$\|y\| < \frac{1}{1-\varepsilon_1} \|x\| ,$$

and hence,

$$\begin{aligned} \|Tx\| &\le \|Ty\| + \|x - y\| < \frac{\varepsilon}{2} \|y\| + \varepsilon_1 \|y\| \\ &= \Big(\frac{\varepsilon}{2} + \varepsilon_1\Big) \|y\| < \frac{1}{1-\varepsilon_1} \Big(\frac{\varepsilon}{2} + \varepsilon_1\Big) \|y\| \overset{\text{by (11.15)}}{=} \varepsilon \|x\|. \end{aligned}$$

To prove (11.14), suppose for contradiction that (11.14) is false. Let $\varepsilon_1 > 0$ so that

$$\big(\forall\, \mathrm{sign}\, x \in L_p(A)\big)\big(\forall y \in K_\varepsilon\big): \quad \|x - y\| \ge \varepsilon_1 \|y\| . \tag{11.16}$$

Let $\lambda = \sup\{\|y\| : y \in K_\varepsilon\}$. Since T is gentle narrow, $\lambda > 0$. Let

$$\eta = \frac{\varepsilon_1 \lambda}{4^{1/p} \mu(A)^{1/p}} \tag{11.17}$$

and observe that

$$\eta < \frac{\varepsilon_1 \mu(A)^{1/p}}{4^{1/p} \mu(A)^{1/p}} < \frac{1}{2} . \tag{11.18}$$

Since φ is p-gentle, there exist $M > 0$ and $\delta_1 > 0$ so that

$$\big(1 - \varphi(M)\big)^2 \ge 1/2 \tag{11.19}$$

and

$$M^{2-p} \big(\varphi(M)\big)^p \le \varepsilon^p \, \frac{p\,(p-1)}{16} \left(\frac{\eta}{1-\eta}\right)^{2-p} - \delta_1 \, \frac{M^2 \varepsilon^p (1 - 2^p \eta^p)}{\eta^p 2^{2p-2} \varepsilon_1^p \lambda^p} . \tag{11.20}$$

Choose $\varepsilon_2 > 0$ and then $\delta_2 > 0$ so that

$$(\lambda - \varepsilon_2)^p + \frac{p\,(p-1)}{2^{6-2p} M^2} \cdot \left(\frac{\eta}{1-\eta}\right)^{2-p} \cdot \frac{\varepsilon_1^p \lambda^p}{1 - 2^p \eta^p} - \delta_2 > \lambda^p \tag{11.21}$$

(by (11.18) the second summand in the left-hand side of the inequality is positive). Let $\delta = \min\{\delta_1, \delta_2\}$. By (11.20) and (11.21), we obtain

$$M^{2-p} \big(\varphi(M)\big)^p \le \varepsilon^p \, \frac{p\,(p-1)}{16} \left(\frac{\eta}{1-\eta}\right)^{2-p} - \delta \, \frac{M^2 \varepsilon^p (1 - 2^p \eta^p)}{\eta^p 2^{2p-2} \varepsilon_1^p \lambda^p} . \tag{11.22}$$

and

$$(\lambda - \varepsilon_2)^p + \frac{p\,(p-1)}{2^{6-2p} M^2} \cdot \left(\frac{\eta}{1-\eta}\right)^{2-p} \cdot \frac{\varepsilon_1^p \lambda^p}{1 - 2^p \eta^p} - \delta > \lambda^p . \tag{11.23}$$

Choose $y \in K_\varepsilon$ with

$$\|y\| > \max\Big\{\frac{\lambda}{2^{1/p}},\, \lambda - \varepsilon_2\Big\}\,. \tag{11.24}$$

Define a sign x on A by

$$x(t) = \begin{cases} 0, & \text{if } |y(t)| \le 1/2, \\ \operatorname{sign}(y), & \text{if } |y(t)| > 1/2, \end{cases}$$

and put

$$B = \{t \in A : \eta \le |y(t)| \le 1 - \eta\}.$$

Since x is a sign on A and $y \in K_\varepsilon$, it follows from (11.16) that

$$\begin{aligned} \varepsilon_1^p \|y\|^p \le \|x - y\|^p &= \int_B |x - y|^p \,\mathrm{d}\mu + \int_{A\setminus B} |x - y|^p \,\mathrm{d}\mu \\ &\le \frac{\mu(B)}{2^p} + \big(\mu(A) - \mu(B)\big)\eta^p\,. \end{aligned}$$

Hence, using (11.24) and (11.17), we deduce that

$$\mu(B)\Big(\frac{1}{2^p} - \eta^p\Big) \ge \varepsilon_1^p\|y\|^p - \mu(A)\eta^p \ge \varepsilon_1^p\,\frac{\lambda^p}{2} - \frac{\varepsilon_1^p\lambda^p}{4} = \varepsilon_1^p\,\frac{\lambda^p}{4}\,,$$

that is,

$$\mu(B) \ge \frac{2^{p-2}\varepsilon_1^p\lambda^p}{1 - 2^p\eta^p}\,. \tag{11.25}$$

In particular, by (11.18) we have that $\mu(B) > 0$. Observe that (11.22) and (11.25) imply

$$\begin{aligned} M^{-p}\big(\varphi(M)\big)^p &\le \varepsilon^p\,\frac{p\,(p-1)}{16M^2}\Big(\frac{\eta}{1-\eta}\Big)^{2-p} - \frac{\varepsilon^p\delta(1 - 2^p\eta^p)}{\eta^p 2^{2p-2}\varepsilon_1^p\lambda^p} \\ &\le \varepsilon^p\,\frac{p\,(p-1)}{16M^2}\Big(\frac{\eta}{1-\eta}\Big)^{2-p} - \frac{\varepsilon^p\delta}{\eta^p 2^p\mu(B)}\,. \end{aligned} \tag{11.26}$$

By Lemma 11.24, we choose $h \in B_{L_\infty(B)}$ so that (a) and (b) hold. By Lemma 7.63(1), there exists a sign number $\theta \in \{-1, 1\}$ so that

$$\|Ty + \theta\eta Th\|^p \le \|Ty\|^p + \eta^p\|Th\|^p\,.$$

Let $z = y + \theta\eta h$. Then by (a) and the choice of $y \in K_\varepsilon$,

$$\begin{aligned} \|Tz\|^p &\le \|Ty\|^p + \eta^p\|Th\|^p \le \varepsilon^p\|y\|^p + \eta^p 2^p\mu(B)M^{-p}\big(\varphi(M)\big)^p \\ &\overset{(11.26)}{\le} \varepsilon^p\|y\|^p + \eta^p 2^p\mu(B)\varepsilon^p\,\frac{p(p-1)}{16M^2}\Big(\frac{\eta}{1-\eta}\Big)^{2-p} - \varepsilon^p\delta. \end{aligned} \tag{11.27}$$

On the one hand, by condition (b) and (11.19), we have

$$\|z\|^p \geq \|y\|^p + \frac{p\,(p-1)}{2^{4-p}M^2} \cdot \frac{\eta^2}{(1-\eta)^{2-p}} \cdot \mu(B) - \delta\,. \tag{11.28}$$

Then (11.27) together with (11.28), give

$$\frac{\|Tz\|^p}{\|z\|^p} \leq \varepsilon^p\,,$$

and this implies that $z \in K_\varepsilon$ (note that $z \in B_{L_\infty(A)}$ by definitions of z and B). On the other hand, we can continue the estimate (11.28) taking into account the choice of y, (11.24) and (11.23) as follows:

$$\|z\|^p > (\lambda - \varepsilon_2)^p + \frac{p\,(p-1)}{2^{4-p}M^2} \cdot \frac{\eta^2}{(1-\eta)^{2-p}} \cdot \mu(B) - \delta$$
$$\overset{\text{by (11.25)}}{\geq} (\lambda - \varepsilon_2)^p + \frac{p\,(p-1)}{2^{4-p}M^2} \cdot \frac{\eta^2}{(1-\eta)^{2-p}} \cdot \frac{2^{p-2}\varepsilon_1^p \lambda^p}{1-2^p\eta^p} - \delta \overset{\text{by (11.23)}}{>} \lambda^p\,.$$

This contradicts the choice of λ. □

Example 7.56 demonstrates that an analog of Theorem 11.20 for $p > 2$ is false.

11.3 C-narrow operators on $C(K)$-spaces

In 1996 V. Kadets and Popov [57] extended the notion of narrow operators to operators defined on $C(K)$-spaces. The definition in this setting needs to be different since $\{1, -1, 0\}$-valued functions are never continuous unless they are constant. V. Kadets and Popov used the idea of Rosenthal's characterization of narrow operators on L_1 (see Theorem 7.30) to generalize them to $C(K)$-spaces. This approach proved quite fruitful as presented in this section. Following a suggestion from [63] here we call these operators C-narrow, even though in [57] they were just called narrow.

Another generalization of the notion of narrow operators to $C(K)$-spaces was introduced by V. Kadets, Shvidkoy and Werner [63] in 2001. Their approach came from the theory of spaces with the Daugavet property, and thus their definition of narrow operators is quite different – it was designed to work on spaces with the Daugavet property, including $C(K)$-spaces and L_1. We do not present the theory of these operators here, since it is really a part of the modern theory of spaces with the Daugavet property, which is broad enough for a monograph of its own, and would take us too far away from our principal subject. For the interested reader, we list here the relevant literature, which continues to grow: [15, 16, 53, 54, 60, 61, 63, 132].

V. Kadets, Shvidkoy and Werner [63] proved that their notion of narrow operators coincides with the notion presented in this section for operators defined on $C(K)$ if K is a perfect compact Hausdorff space. They also showed that every narrow operator on L_1 (in the usual sense of Definition 1.5) is narrow in the sense of [63], and they asked whether these two notions coincide for operators on L_1.

For simplicity of the notation, we consider the space $C[0,1]$, however the same notions and results hold for $C(K)$, where K is any compact set without isolated points.

In this section all spaces are considered over the reals.

The definition of C-narrow operators on $C[0,1]$

Given a Banach space X, an operator $T \in \mathcal{L}(C[0,1], X)$, and a segment $I = [a,b] \subseteq [0,1]$, by $C^0[a,b] = C^0(I)$ we denote the subspace of $C[0,1]$ consisting of all functions vanishing off I, and by T_I we denote the restriction of T to $C^0(I)$.

Definition 11.25. Let X be a Banach space. An operator $T \in \mathcal{L}(C[0,1], X)$ is called *C-narrow* if for every $I = [a,b] \subseteq [0,1]$ the operator T_I is not an into isomorphism.

In other words, T is C-narrow provided for every $I = [a,b] \subseteq [0,1]$ and every $\varepsilon > 0$ there exists $f \in S_{C^0(I)}$ such that $\|Tf\| < \varepsilon$.

The following two propositions describe the relationships between C-narrow and $C[0,1]$-singular operators.

Proposition 11.26. *Let X be a Banach space. Then every $C[0,1]$-singular operator $T \in \mathcal{L}(C[0,1], X)$ is C-narrow.*

To prove Proposition 11.26, it is enough to observe that $C^0(I)$ is isomorphic to $C[0,1]$ whenever $a < b$.

Proposition 11.27. *There exists a C-narrow projection $T \in \mathcal{L}(C[0,1])$ that fixes a copy of $C[0,1]$.*

Proof. Let Δ be the Cantor set on $[0,1]$. We define an operator $T \in \mathcal{L}(C[0,1])$ by setting $(Tf)(t) = f(t)$ for $t \in \Delta$, and extending Tf to any constituent interval (α,β) of $[0,1] \setminus \Delta$ by linear extrapolation

$$(Tf)(t) = \frac{t-\beta}{\alpha-\beta} f(\alpha) + \frac{t-\alpha}{\beta-\alpha} f(\beta)\,.$$

Observe that T is a projection of $C[0,1]$ onto the subspace X of $C[0,1]$ of all functions that are linear on any constituent interval (α,β) of $[0,1] \setminus \Delta$, and that the restriction $f|_\Delta$ of an element $f \in X$ to the Cantor set is an isometry between X and $C(\Delta)$. By the Milyutin theorem [107], $C(\Delta)$ is isomorphic to $C[0,1]$. Hence, X is isomorphic to $C[0,1]$. Since $T|_X$ is an isometry, T fixes a copy of $C[0,1]$. To show that T is C-narrow, observe that for any interval $I = [a,b] \subseteq [0,1]$ with $a < b$ there exists a constituent interval $(\alpha,\beta) \subseteq [a,b]$ of $[0,1] \setminus \Delta$, on which $T_{[\alpha,\beta]} = 0$. □

The following statement asserts that in the definition of a C-narrow operator one may claim the existence of a positive function at which the operator is small.

Proposition 11.28. *Let X be a Banach space. An operator $T \in \mathcal{L}(C[0,1], X)$ is C-narrow if and only if for every $I = [a,b] \subseteq [0,1]$ with $a < b$ and every $\varepsilon > 0$ there exists $f \in S_{C^0(I)}$ such that $f \geq 0$ and $\|Tf\| < \varepsilon$.*

Proof. By the definition of a C-narrow operator, we only need to prove one implication. Fix any $I = I_0 = [a,b] \subseteq [0,1]$ with $a < b$ and $\varepsilon > 0$. We choose $n \in \mathbb{N}$ so that

$$(1-\varepsilon)^{-1}\Big(\frac{\varepsilon}{2} + \frac{\|T\|}{n}\Big) < \varepsilon\,, \tag{11.29}$$

and $f_1 \in S_{C^0(I_0)}$ so that $\|Tf_1\| < \varepsilon/2$. Without loss of generality, we may and do assume that $\max_{t \in I_0} f_1(t) = 1$ (otherwise we multiply f_1 by -1). Choose a nontrivial interval $I_1 \subseteq I_0$ such that $f_1(t) > 1-\varepsilon$, for all $t \in I_1$. Next we choose $f_2 \in S_{C^0(I_1)}$ so that $\|Tf_2\| < \varepsilon/2$ and $\max_{t \in I_1} f_2(t) = 1$, and a nontrivial interval $I_2 \subseteq I_1$ so that $f_2(t) > 1-\varepsilon$ for all $t \in I_2$. Likewise, for each $k = 2,\dots,n-1$, we choose $f_{k+1} \in S_{C^0(I_k)}$ so that $\|Tf_{k+1}\| < \varepsilon/2$ and $\max_{t \in I_k} f_{k+1}(t) = 1$, and a nontrivial interval $I_{k+1} \subseteq I_k$ so that $f_{k+1}(t) > 1-\varepsilon$, for all $t \in I_{k+1}$. Let $g = \frac{1}{n}\sum_{k=1}^{n} f_k$, and note that $g \in S_{C^0(I)}$, $\|Tg\| \leq \varepsilon/2$, and $g(t) \geq 1-\varepsilon$, for all $t \in I_n$, and hence, $\|g\| \geq 1-\varepsilon$. We claim that $g \geq -1/n$. Indeed, by definition, g vanishes outside I, and $g \geq 1-\varepsilon > -1/n$ on I_n. Let $0 \leq k \leq n-1$. Thus for all $t \in I_k \setminus I_{k+1}$ we have

$$g(t) = \frac{1}{n}\Big(\sum_{i=1}^{k} g_i(t) + g_{k+1}(t)\Big) \geq \frac{1}{n}\big(k(1-\varepsilon) - 1\big) \geq -\frac{1}{n}\,.$$

This proves the claim, because $I = I_n \cup \bigcup_{k=0}^{n-1}(I_k \setminus I_{k+1})$.

Let

$$g^+ = \frac{g+|g|}{2} \quad \text{and} \quad f = \frac{g^+}{\|g^+\|}\,.$$

Since $g \geq 1-\varepsilon$ on I_n, we have that $\|g^+\| = \|g\| \in [1-\varepsilon, 1]$, and since $g \geq -1/n$, one has $\|g - g^+\| \leq 1/n$. Hence,

$$\|Tf\| = \frac{\|Tg^+\|}{\|g^+\|} \leq \frac{1}{1-\varepsilon}\Big(\|Tg\| + \frac{\|T\|}{n}\Big) \leq (1-\varepsilon)^{-1}\Big(\frac{\varepsilon}{2} + \frac{\|T\|}{n}\Big) \overset{\text{by (11.29)}}{<} \varepsilon\,.$$

It remains to observe that $f \geq 0$ and $f \in S_{C^0(I)}$. □

Vanishing points of an operator and ideal properties of C-narrow operators

We give a characterization of C-narrow operators on $C[0,1]$.

Definition 11.29. Let X be a Banach space. We say that an operator $T : C[0,1] \to X$ *vanishes* at a point $t \in [0,1]$ (and write $t \in \operatorname{van} T$) if there exists a sequence of intervals $I_n = [a_n, b_n] \subseteq [0,1]$ with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = t$, and a sequence of nonnegative functions $f_n \in S_{C^0(I_n)}$, converging pointwise to

$$\delta_t(\tau) = \begin{cases} 1, & \text{if } \tau = t, \\ 0, & \text{if } \tau \neq t, \end{cases}$$

and such that $\lim_{n\to\infty} \|Tf_n\| = 0$.

Proposition 11.30. *Let X be a Banach space and $T \in \mathcal{L}(C[0,1], X)$. Then T is C-narrow if and only if* van T *is dense in* $[0,1]$.

Proof. Let T be C-narrow, and $I = I_0 = [a,b] \subseteq [0,1]$ be any segment with $a < b$. Using Proposition 11.28, we construct a nested sequence of segments $I_{n+1} \subseteq I_n$ and a sequence of nonnegative functions $f_n \in S_{C^0(I_n)}$ such that $f_n(t) \geq 1 - 1/n$ for all $t \in I_{n+1}$, and $\|Tf_n\| \leq 1/n$. Without loss of generality, we assume that $\lim_{n\to\infty} \mu(I_n) = 0$. Then the point t such that $\{t\} = \bigcap_{n=1}^{\infty} I_n$ belongs to van T. By arbitrariness of I, van T is dense in $[0,1]$.

The converse implication is immediate. □

Proposition 11.31. *Let X be a Banach space and $T \in \mathcal{L}(C[0,1], X)$. Then the set* van T *is a G_δ-set in* $[0,1]$.

Proof. Let T be given. For each $n \in \mathbb{N}$, we define a set $\mathcal{D}_n \subseteq S_{C[0,1]}$ as follows:

$$\mathcal{D}_n = \left\{ f \in S_{C[0,1]} : (f \geq 0) \& \left(\|Tf\| < \frac{1}{n}\right) \& \left(\operatorname{diam} \operatorname{supp} f < \frac{1}{n}\right) \right\},$$

and for any $f \in S_{C[0,1]}$ let

$$F_n(f) = \left\{ t \in [0,1] : f(t) > 1 - \frac{1}{n} \right\} \text{ and } \mathcal{F}_n = \bigcup_{f \in \mathcal{D}_n} F_n(f).$$

Observe that all F_n are open in $[0,1]$, and van $T \subseteq F_n$ for each n. Hence, the set $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$ is a G_δ-set containing van T. We show the converse inclusion. Let $t \in \mathcal{F}$, that is, $t \in \mathcal{F}_n$ for all n. Hence, for each $n \in \mathcal{N}$ there is $f_n \in \mathcal{D}_n$ with $f_n \geq 1 - 1/n$. Thus, the sequence (f_n) together with the sequence of segments $I_n = [\inf(\operatorname{supp} f_n), \sup(\operatorname{supp} f_n)]$ satisfy Definition 11.29. Thus, van $T = \mathcal{F}$. □

The following statement characterizes vanishing points of an operator.

Proposition 11.32. *Let X be a Banach space, $T \in \mathcal{L}(C[0,1], X)$ and $t \in [0,1]$. Then $t \in$* van T *if and only if for any $x^* \in X^*$, the point t is not an atom of the measure corresponding to T^*x^*.*

Proof. Suppose that $t \in \operatorname{van} T$ with functions (f_n) that appear in Definition 11.29. Let $x^* \in X^*$ and ν be the measure corresponding to T^*x^*, i.e. for each $f \in C[0,1]$,

$$\langle T^*x^*, f\rangle = \int_{[0,1]} f \, \mathrm{d}\nu \, .$$

Then t is not an atom for ν, since

$$\begin{aligned}\nu\big(\{t\}\big) &= \Big|\int_{[0,1]} \delta_t(\tau)\,\mathrm{d}\nu(\tau)\Big| = \Big|\lim_{n\to\infty}\int_{[0,1]} f_n \,\mathrm{d}\nu\Big| \\ &= \Big|\lim_{n\to\infty} x^*(Tf_n)\Big| \le \|x^*\| \lim_{n\to\infty} \|Tf_n\| = 0\,.\end{aligned}$$

For the other direction, let $I_n = [t - 1/n, t + 1/n]$ and $g_n \in S_{C^0(I_n)}$ be any sequence with $g_n \geq 0$ and $g_n(t) = 1$. Since t is not an atom of the measure corresponding to T^*x^* for any $x^* \in X^*$, (g_n) is weakly null. By Mazur's theorem, there is a sequence $f_n \in \operatorname{conv}\{g_k : k \geq n\}$ such that $\lim_{n\to\infty} \|Tf_n\| = 0$. Then sequences (I_n) and (f_n) satisfy Definition 11.29 for t. □

Note that in Proposition 11.32 it suffices to consider only extreme points x^* of B_{X^*}.

Now we are ready to show that the sum of two C-narrow operators defined on $C[0,1]$ is C-narrow.

Theorem 11.33. *Let X be a Banach space. The set of all C-narrow operators from $C[0,1]$ to X is a (norm closed) subspace of $\mathcal{L}(C[0,1], X)$.*

Proof. It is enough to prove that the sum of two C-narrow operators is C-narrow (the rest of the properties are obvious). Let $S, T \in \mathcal{L}(C[0,1], X)$ be narrow. Note that if two measures have no atom at a point t then neither does the sum of these measures. Hence, by Proposition 11.32, $\operatorname{van} S \cap \operatorname{van} T \subseteq \operatorname{van}(S + T)$. By Proposition 11.31, $\operatorname{van} S$ and $\operatorname{van} T$ are dense G_δ-sets in $[0,1]$. By the Baire category theorem, $\operatorname{van} S \cap \operatorname{van} T$ is dense in $[0,1]$, and thus, so is $\operatorname{van}(S + T)$. By Proposition 11.30, $S + T$ is C-narrow. □

The remaining ideal properties are similar to those of narrow operators on Köthe–Banach spaces.

Proposition 11.34. *Let X, Y are Banach spaces. If an operator $T \in \mathcal{L}(C[0,1], X)$ is C-narrow then for every $S \in \mathcal{L}(X, Y)$ the composition operator $S \circ T$ is C-narrow. On the other hand, there exists a bounded operator $S \in \mathcal{L}(C[0,1])$ and a C-narrow operator $S \in \mathcal{L}(C[0,1])$ such that the composition $T \circ S$ is not C-narrow.*

Proof. The first part follows easily from the definition. Let T be the operator from Proposition 11.27, and X be a subspace of $C[0,1]$ isomorphic to $C[0,1]$ such that $T|_X$ is an into isomorphism. Let $S : C[0,1] \to X$ be an isomorphism. Then $T \circ S$ is an into isomorphism, and hence, is not C-narrow. □

C-rich subspaces of $C[0,1]$

In analogy with Köthe–Banach spaces, a subspace X of $C[0,1]$ is called *C-rich* if the quotient map $\tau : C[0,1] \to C[0,1]/X$ is narrow. By Proposition 11.28, we can equivalently say that *a subspace X of $C[0,1]$ is C-rich if and only if for every $I = [a,b] \subseteq [0,1]$ with $a < b$ and every $\varepsilon > 0$ there exist $f \in S_{C^0(I)}$ with $f \geq 0$ and $g \in X$ such that $\|f - g\| < \varepsilon$.* Of course, a complemented subspace X of $C[0,1]$ is C-rich if and only if any projection P of $C[0,1]$ with $\ker P = X$ is C-narrow (equivalently, there exists a C-narrow projection P of $C[0,1]$ with $\ker P = X$).

It is easy to see from the definition that if X is a C-rich subspace of $C[0,1]$ then any subspace Y of $C[0,1]$ with $Y \supseteq X$ is C-rich as well. On the other hand, distinct C-rich subspaces may have trivial intersection. In analogy with Proposition 5.4, one can construct quasi-complemented C-rich subspaces.

Proposition 11.35. *There exists a pair of C-rich quasi-complemented subspaces of $C[0,1]$.*

Proof. We split the Schauder system in $C[0,1]$ into two subsequences (f_n) and (g_n), attributing the functions vanishing precisely off some interval of length 2^{-2n+1} to the first subsequence, and those vanishing precisely off some interval of length 2^{-2n} to the second one. Then $[f_n]$ and $[g_n]$ are quasi-complemented C-rich subspaces. □

Nevertheless, Theorem 11.33 yields that $C[0,1]$ cannot be decomposed into a direct sum of C-rich subspaces. Indeed, otherwise the identity (which is not C-narrow) would equal the sum of two C-narrow projections, which contradicts Theorem 11.33.

Proposition 11.36. *There is a decomposition $C[0,1] = X \oplus Y$ into subspaces isomorphic to $C[0,1]$ with Y C-rich.*

Proof. Let T be the narrow projection of $C[0,1]$ onto X from Proposition 11.27 with X isomorphic to $C[0,1]$, and let $Y = \ker T$. Since Y evidently contains a subspace isomorphic to $C[0,1]$ (for instance, $C^0[1/3, 2/3]$), by Pełczyński's decomposition method [108], Y is itself isomorphic to $C[0,1]$. □

Corollary 11.37. *Let a subspace Z of $C[0,1]$ be isomorphic to a direct sum $C[0,1] \oplus E$, where E is a separable Banach space. Then Z is isomorphic to a C-rich subspace of $C[0,1]$.*

Proof. Note that in the decomposition $C[0,1] = X \oplus Y$ given by Proposition 11.36, the space X is isomorphic to $C[0,1]$, and thus, is universal. If $F \subseteq X$ is a subspace isomorphic to E then $F \oplus Y$ is a C-rich subspace of $C[0,1]$ isomorphic to Z. □

The converse assertion is also true: every C-rich subspace of $C[0,1]$ is isomorphic to a direct sum $C[0,1] \oplus E$ for a suitable separable Banach space E. Equivalently speaking, every C-rich subspace of $C[0,1]$ contains a complemented copy of $C[0,1]$. The proof is not so obvious (see [57]).

The Daugavet property of C-rich subspaces of $C[0,1]$

Definition 11.38. Let X be a C-rich subspace of $C[0,1]$ and Y be a Banach space. An operator $T \in \mathcal{L}(X,Y)$ is called *C-narrow on X* if for each $\varepsilon > 0$ and each $I = [a,b] \subseteq [0,1]$ with $a < b$ there exist functions $f \in S_{C^0(I)}$ and $g \in X$ such that $\|f - g\| < \varepsilon$ and $\|Tf\| < \varepsilon$.

Definition 11.38 is consistent with Definition 11.25 for $X = C[0,1]$. Without essential changes, one can prove the above results on C-narrow operators defined on $C[0,1]$ for the setting of operators defined on C-rich subspaces. In particular, one can claim that $f \geq 0$ in Definition 11.38.

Theorem 11.39. *Every C-rich subspace X of $C[0,1]$ has the Daugavet property for C-narrow operators on X.*

Proof. Let $T \in \mathcal{L}(X)$ be a C-narrow operator and $\varepsilon > 0$. Let $f \in S_X$ with $\|Tf\| \geq \|T\| - \varepsilon$, and $g = Tf$. Without loss of generality we assume that $\|g\| = g(t_0)$ for some $t_0 \in [0,1]$. Let $I \subseteq [0,1]$ be any nontrivial interval such that $g(t) > \|T\| - \varepsilon$ for each $t \in I$. Since T is C-narrow, there exists $s \in S_{C^0(I)}$ with $s \geq 0$ and $h \in S_X$ such that $\|s - h\| < \varepsilon/10$ and $\|Th\| < \varepsilon$. Let $u_\lambda = (1-\varepsilon)f + \lambda h$. Then $\|u_\lambda\|$ is a continuous function of λ. Since $\|u_0\| = 1 - \varepsilon$ and $\|u_2\| = 2\|h\| - (1-\varepsilon) > 1$, there is $\lambda \in (0,2)$ so that $\|u_\lambda\| = 1$. Let $u = u_\lambda$. If $t \in [0,1] \setminus I$, then

$$|u(t)| = |(1-\varepsilon)f(t) + \lambda_0(h-s)(t) + \lambda_0 s(t)| \leq (1-\varepsilon) + \lambda_0 \frac{\varepsilon}{10} < 1\,,$$

and for all $t \in [0,1]$ we have $u(t) \geq (-1-\varepsilon) - \lambda_0 \frac{\varepsilon}{10} > -1$. Thus, the function u attains its norm at some point $\tau \in I$, where $u(\tau) = 1$. We estimate $\|I + T\|$ as follows:

$$\begin{aligned} \|I + T\| &\geq \|(I+T)u\| = \|u + (1-\varepsilon)Tf + \lambda_0 Th\| \\ &\geq \|u + Tf\| - \varepsilon\|Tf\| - \lambda_0\|Th\| \\ &\geq u(\tau) + g(\tau) - \varepsilon\|T\| - \varepsilon\lambda_0 \\ &\geq 1 + \|T\| - \varepsilon(\|T\| + 3)\,. \end{aligned}$$

By arbitrariness of $\varepsilon > 0$, the theorem is proved. □

Corollary 11.40. *$C[0,1]$ has the Daugavet property for the set $\mathcal{N}$ of all C-narrow operators on $C[0,1]$.*

Foiaş and Singer [41] proved that $C[0,1]$ has the Daugavet property for the following class $\mathcal{M}$ of operators: $T \in \mathcal{M} \subset C[0,1]$ if and only if for every $[\alpha,\beta] \subseteq [0,1]$

with $\alpha < \beta$ and every $\varepsilon > 0$ there exists $I = [a, b] \subseteq [\alpha, \beta]$ with $a < b$ such that $\|T_I\| < \varepsilon$. Evidently, $\mathcal{M} \subseteq \mathcal{N}$. However, the converse is not true. Indeed, let (e_n) be a sequence in $C[0,1]$ equivalent to the unit vector basis of c_0, and let (f_n) and (g_n) be the subsequences of the Schauder system in $C[0,1]$ constructed in the proof of Proposition 11.35. One can show that the operator $T \in \mathcal{L}(C[0,1])$ which extends by linearity and continuity the equalities $Tf_k = e_k$ and $Tg_k = 0$ for each $k \in \mathbb{N}$ is C-narrow and does not belong to $\mathcal{M}$.

The following statement, which is the main result of this subsection, is a consequence of Corollary 11.37 and Theorem 11.39.

Corollary 11.41. *Let X be a Banach space isomorphic to a direct sum $C[0,1] \oplus E$, where E is a separable Banach space. Then there exists an equivalent norm on X with respect to which X has the Daugavet property for $C[0,1]$-singular operators.*

The following statement is a consequence of Corollary 11.40 and Proposition 11.26.

Corollary 11.42. *$C[0,1]$ has the Daugavet property for $C[0,1]$-singular operators.*

Independently (however, somewhat later) the same result was obtained in [140].

The well-known fact that $C[0,1]$ has no unconditional basis can be interpreted that $C[0,1]$ cannot be decomposed into an unconditional sum of one-dimensional subspaces. Using the Daugavet property, we can claim more.

Theorem 11.43. *If $C[0,1]$ is decomposed into an unconditional sum of subspaces then, at least, one of them is isomorphic to $C[0,1]$.*

For the proof we need the following lemma.

Lemma 11.44. *Let X be a Banach space and let $\mathcal{M} \subseteq \mathcal{L}(X)$ be a linear subspace. If X has the Daugavet property for $\mathcal{M}$ then the identity of X cannot be represented as a pointwise unconditionally convergent series of operators from $\mathcal{M}$.*

Proof. Assume on the contrary that $I = \sum_{n=1}^{\infty} T_n$ is a pointwise unconditionally convergent series with $T_n \in \mathcal{M}$ for each $n \in \mathcal{N}$. Then the Uniform Boundedness Principle implies that

$$\mathcal{D} \stackrel{\text{def}}{=} \sup_{I \in \mathbb{N}^{<\omega}} \Big\| \sum_{n \in I} T_n \Big\| < \infty .$$

By Theorem 11.33, for each $I \in \mathbb{N}^{<\omega}$ the operator $\sum_{n \in I} T_n$ is C-narrow. Thus, using the Daugavet equation, we obtain

$$\begin{aligned} 1 + \mathcal{D} &= \sup_{I \in \mathbb{N}^{<\omega}} \Big\| I - \sum_{n \in I} T_n \Big\| = \sup_{I \in \mathbb{N}^{<\omega}} \Big\| \sum_{n \in \mathbb{N} \setminus I} T_n \Big\| \\ &\le \sup_{I \in \mathbb{N}^{<\omega}} \sup_{J \in \mathbb{N}^{<\omega}, J \cap I = \emptyset} \Big\| \sum_{n \in J} T_n \Big\| \le \sup_{J \in \mathbb{N}^{<\omega}} \Big\| \sum_{n \in J} T_n \Big\| = \mathcal{D} , \end{aligned}$$

which is a contradiction. □

Proof of Theorem 11.43. Let $C[0,1] = \bigoplus \sum_{n=1}^{\infty} X_n$ be an unconditional decomposition into subspaces, and let P_n be the corresponding projection of $C[0,1]$ onto X_n parallel to all other X_m. Then $I = \sum_{n=1}^{\infty} P_n$ is a pointwise unconditionally convergent series. By Corollary 11.42 and Lemma 11.44, there is $n \in \mathbb{N}$ such that P_n fixes a copy of $C[0,1]$. Thus X_n contains an isomorph of $C[0,1]$. Since X_n is complemented, by Pełczyński's decomposition method [108], X_n is isomorphic to $C[0,1]$. □

We remark that using the same method and Lemma 11.44, one can prove the same result for L_1, which is due to Enflo and Starbird [37], that is, if L_1 is decomposed into an unconditional sum of subspaces then, at least, one of them is isomorphic to L_1.

Open problem 11.45. Is the subspace of $\mathcal{L}(C[0,1])$ consisting of all C-narrow operators, complemented in $\mathcal{L}(C[0,1])$?

11.4 Narrow operators on $L_\infty(\mu)$-spaces

In this section we present the current status of the theory of narrow operators defined on L_∞. Here we use the standard definition of narrow operators, i.e. Definition 1.5. We note that we do not know whether this is equivalent to Definition 10.1 for operators on L_∞ (see Open problem 10.3). Majority of results about narrow operators rely on the assumption that the domain space has an absolutely continuous norm. Since L_∞ fails this condition, new techniques are needed for this setting. The theory of narrow operators on L_∞ is quite different from the theory of narrow operators on absolutely continuous spaces and there are many unresolved questions. It is not even known whether in the definition of narrow operators the mean zero condition can be omitted, like in the absolutely continuous case (see Open problem 1.10). Moreover, not every compact operator on L_∞ is narrow and there exist nonnarrow bounded linear functionals. We present here several surprising examples and the theory of order-to-norm continuous narrow operators on L_∞, as well as several natural open problems. The results come from [71, 72] and [93]. We note that we presented a few initial results about narrow operators defined on L_∞ in Chapter 10, Sections 10.1 and 10.2.

Nonnarrow continuous linear functionals

In Chapter 10 we saw an example of a nonnarrow continuous linear functional on L_∞ (Example 10.12). V. Kadets has communicated to us an idea of the following example.

Example 11.46. Let $f \in L_\infty^*$ be a non-zero multiplicative functional. Then f is nonnarrow.

Here we consider L_∞ as a Banach algebra. A functional $f : B \to \mathbb{R}$ on a Banach algebra B is called *multiplicative* if $f(x \cdot y) = f(x)f(y)$ for all $x, y \in B$. A functional $f \in L_\infty^*$ is multiplicative if and only if its kernel $\ker f$ is a maximal ideal in L_∞.

Proof. Since $f \neq 0$, there exists $A \in \Sigma^+$ such that $f(\mathbf{1}_A) \neq 0$. Thus for every sign x on A we have $0 \neq f(\mathbf{1}_A) = f(x \cdot x) = (f(x))^2$. Hence f is not narrow. □

The following question was posed in [117].

Open problem 11.47. Does there exist a narrow functional $f \in L_\infty^*$ which is not strictly narrow?

Rich subspaces of L_∞

The following is an analog of Theorem 2.8, which identifies conditions for the subspace X to be rich. Here, unlike in Theorem 2.8, we only obtain a sufficient condition for X to be strictly rich.

Theorem 11.48. *Let X be a weak*-closed subspace of L_∞. Then X is rich if and only if $X \cap L_\infty(A) \neq \{0\}$ for every $A \in \Sigma^+$.*

Proof. The necessity of the condition is obvious. Let us prove its sufficiency. Assume that $X \cap L_\infty(A) \neq \{0\}$ for every $A \in \Sigma^+$. For each $A \in \Sigma^+$ we consider the set

$$K_A = B_{L_\infty^0(A)} \cap X .$$

Since the Lebesgue measure is atomless, we have that $\dim(X \cap L_\infty(A)) = \infty$. Since $\dim L_\infty(A)/L_\infty^0(A) = 1 < \infty$, we obtain that $X \cap L_\infty^0(A) \neq \{0\}$, by Lemma 2.9, $K_A \neq \{0\}$. Since K_A is a subset of a weak*-closed set, the Banach–Alaoglu theorem implies that K_A is compact in the weak* topology of the space L_∞. By convexity of K_A, the Krein–Milman theorem yields that K_A has an extreme point $x \in K_A$. It follows from a standard argument that $|x| = \mathbf{1}_A$, see, for example, the proof of Theorem 2.8. □

Corollary 11.49. *Suppose that the kernel* $\ker f$ *of a functional $f \in L_\infty^*$ is weak*-closed in L_∞. Then f is strictly rich.*

Order structure and order-to-norm continuous operators on L_∞

As we already know, a compact operator defined on L_∞ need not be narrow. However, for every Banach space X every order-to-norm continuous AM-compact operator $T \in \mathcal{L}(L_\infty(\mu), X)$ is narrow (see Theorem 10.17 and Corollary 10.18). The proof of Theorem 10.17 is very involved, however for the special case of L_∞ we can now give a much shorter proof of Corollary 10.18. Most of the results of this subsection were obtained in [72].

Theorem 11.50. *Let* (Ω, Σ, μ) *be a finite atomless measure space, and let* X *be a Banach space. Then every AM-compact order-to-norm continuous* $T \in \mathcal{L}(L_\infty(\mu), X)$ *is narrow.*

It is worth mentioning that the order boundedness and the norm boundedness for sets in $L_\infty(\mu)$ coincide. Thus, when speaking of bounded sequences in $L_\infty(\mu)$, we need not specify the kind of boundedness we mean.

Proof of Theorem 11.50. Let $T \in \mathcal{L}(L_\infty(\mu), X)$ be AM-compact and order-to-norm continuous. Fix any $A \in \Sigma$ and $\varepsilon > 0$. Consider a Rademacher-type system (r_n) on $L_\infty(A)$. Since (r_n) is order bounded, (Tr_n) is relatively compact in X. Hence, there are indices $n \neq m$ such that for $x_1 = (r_n - r_m)/2$ we have $|x_1| = \mathbf{1}_{B_1}$, $B_1 \subset A$, $\mu(B_1) = \mu(A)/2$, $\int_\Omega x_1 \, \mathrm{d}\mu = 0$ and $\|Tx_1\| < \varepsilon/2$. Setting $A_1 = A \setminus B_1$ we do the same with the set A_1 instead of A to find $x_2 \in L_\infty(\mu)$ with $|x_2| = \mathbf{1}_{B_2}$, $B_2 \subset A_1$, $\mu(B_2) = \mu(A_1)/2 = \mu(A)/4$, $\int_\Omega x_2 \, \mathrm{d}\mu = 0$ and $\|Tx_2\| < \varepsilon/4$. Continuing this procedure, we construct a sequence (x_n) in $L_\infty(\mu)$ such that $|x_n| = \mathbf{1}_{B_n}$, $A = \bigsqcup_{n=1}^\infty B_n$ (up to a set of measure zero), $\int_\Omega x_n \, \mathrm{d}\mu = 0$ and $\|Tx_n\| < \varepsilon/2^n$. We set $x(\omega) = x_n(\omega)$ for $\omega \in B_n$ and $x(\omega) = 0$ for $\omega \in \Omega \setminus A$, and observe that $|x| = \mathbf{1}_A$, $\int_\Omega x \, \mathrm{d}\mu = 0$ and $\sum_{k=1}^n x_k \xrightarrow{\mathrm{o}} x$ in $L_\infty(\mu)$, by Proposition 1.18. By the order-to-norm continuity of T, the last condition implies that $\lim_{n\to\infty} \|T(\sum_{k=1}^n x_k)\| = \|Tx\|$. And since $\|T(\sum_{k=1}^n x_k)\| \le \sum_{k=1}^n \|Tx_k\| < \varepsilon$, we obtain $\|Tx\| \le \varepsilon$. □

So, the following question arises naturally:

> *Does there exist a Banach space* X *such that every order-to-norm continuous operator* $T \in \mathcal{L}(L_\infty, X)$ *is narrow while not every operator of this kind is AM-compact?*

We will show that the answer is positive.

Our next goal is to give some characterizations of order-to-norm continuity for operators with the domain $L_\infty(\mu)$.

Theorem 11.51. *Let* (Ω, Σ, μ) *be a finite atomless measure space,* X *be a Banach space, and* $T \in \mathcal{L}(L_\infty(\mu), X)$. *Then the following conditions are equivalent:*

(i) T *is order-to-norm continuous.*

(ii) T *is order-to-norm* σ*-continuous.*

(iii) *For any bounded sequence* (x_n) *in* $L_\infty(\mu)$ *converging to zero in measure, one has that* $\|Tx_n\| \to 0$.

(iv) *Let* $p \in [1, \infty)$. *For any bounded sequence* (x_n) *in* $L_\infty(\mu)$, *the condition* $\|x_n\|_p \to 0$ *implies that* $\|Tx_n\| \to 0$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Let (x_n) be a bounded sequence in $L_\infty(\mu)$ converging to zero in measure. Suppose, for the sake of contradiction, that $\|Tx_n\| \geq \delta > 0$ for infinitely many $n \in \mathbb{N}$. Without loss of generality we assume that this is true for all $n \in \mathbb{N}$. Passing to a subsequence, we obtain that $x_{n_k} \to 0$ a.e. and still $\|Tx_{n_k}\| \geq \delta > 0$, which contradicts (ii).

(iii) $\Rightarrow$ (iv) It is enough to note that $\|x_n\|_p \to 0$ implies that $x_n \to 0$ in measure.

(iv) $\Rightarrow$ (ii) Suppose that $x_n \xrightarrow{\text{o}} 0$. Then $|x_n| \leq y_n \downarrow 0$ for some sequence (y_n) in $L_\infty(\mu)$. By Proposition 1.18, $y_n \to 0$ a.e., and therefore $\|y_n\|_p \to 0$. Since $\|x_n\|_p \leq \|y_n\|_p$ for each n, we obtain that $\|x_n\|_p \to 0$. By (iv), $\|Tx_n\| \to 0$.

(ii) $\Rightarrow$ (i) First we prove the following statement.

Claim. Let $x_\alpha \downarrow 0$ in $L_\infty(\mu)$. Then there exists a strictly increasing sequence of indices (α_n) such that $\inf_n x_{\beta_n} = 0$ for any sequence of indices $\beta_n \geq \alpha_n$.

Indeed, since $0 \leq x_{\beta_n} \leq x_{\alpha_n}$, it is enough to show that $\inf_n x_{\alpha_n} = 0$. Set $t_\alpha = \int_\Omega x_\alpha \, d\mu$ and observe that there exists $t_0 = \lim_\alpha t_\alpha$ because (t_α) is decreasing and bounded below by 0. Choose a strictly increasing sequence of indices (α_n) so that $\lim_{n\to\infty} t_{\alpha_n} = t_0$. Since the sequence (x_{α_n}) is decreasing and bounded below by 0, $z(\omega) = \lim_{n\to\infty} x_{\alpha_n}(\omega) \geq 0$ exists a.e. Observe that $z = \inf_n x_{\alpha_n}$ and, by the Lebesgue theorem, $t_0 = \int_\Omega z \, d\mu$. To prove the claim, it is sufficient to show that $t_0 = 0$. Suppose otherwise that $t_0 > 0$. Since $\inf_\alpha x_\alpha = 0$ and $z \neq 0$, there exists an index β such that $z \wedge x_\beta < z$. For every $n \in \mathbb{N}$ we choose an index γ_n so that $\gamma_n \geq \beta$ and $\gamma_n \geq \alpha_n$. Then $y = \inf_n x_{\gamma_n} < z$ and $\int_\Omega y \, d\mu = \lim_{n\to\infty} t_{\gamma_n} < t_0$, which contradicts the choice of t_0. Thus, the claim is proved.

Let T be order-to-norm σ-continuous. It is enough to prove that T is order-to-norm continuous at zero. Suppose that a net (x_α) order converges to 0, i.e. there is a net (u_α) such that $|x_\alpha| \leq u_\alpha \downarrow 0$. By the claim, there exists a strictly increasing sequence of indices (α_n) with $\inf_n u_{\alpha_n} = 0$.

Fix any $\varepsilon > 0$ and consider the index set $A_\varepsilon = \{\alpha : \|Tx_\alpha\| \geq \varepsilon\}$. We show that the set A_ε is bounded from above, that is, there exists a β such that $\alpha < \beta$ for each $\alpha \in A_\varepsilon$. Indeed, supposing the contrary, we obtain that there exists a sequence (β_n) of indices $\beta_n \in A_\varepsilon$ such that $\beta_n > \alpha_n$ for each n. Then $|x_{\beta_n}| \leq u_{\beta_n} \leq u_{\alpha_n}$ and $\inf_n u_{\alpha_n} = 0$, which implies that $x_{\beta_n} \xrightarrow{\text{o}} 0$. However, we have that $\|Tx_{\beta_n}\| \geq \varepsilon$ which contradicts the order-to-norm σ-continuity of T at zero, which is another contradiction. Thus, the set A_ε is bounded above by some β and hence, $\alpha \notin A_\varepsilon$ (equivalently, $\|Tx_\alpha\| < \varepsilon$) for every $\alpha \geq \beta$. This means that T is order-to-norm continuous at zero. □

By Theorem 11.51, if an operator $T \in \mathcal{L}(L_\infty(\mu), X)$ can be continuously extended to $L_p(\mu)$ for some $p < \infty$, then T is order-to-norm continuous. We will show below that the converse is not valid: not every order-to-norm continuous operator from $\mathcal{L}(L_\infty(\mu), c_0(\Gamma))$ can be extended to $L_p(\mu)$ for some $1 \leq p < \infty$ (see Example 11.57).

Here is another characterization of order-to-norm continuity for operators from $L_\infty(\mu)$ to X.

Lemma 11.52. *Let (Ω, Σ, μ) be a finite atomless measure space, and X be a Banach space. An operator $T \in \mathcal{L}(L_\infty(\mu), X)$ is order-to-norm continuous if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in L_\infty(\mu)$ with $\|x\| \le 1$ and $\mu(\operatorname{supp} x) < \delta$ we have $\|Tx\| < \varepsilon$.*

Proof. For the "only if" part, suppose to the contrary that for some $\varepsilon > 0$ there exists a sequence $x_n \in L_\infty(\mu)$, $\|x_n\| \le 1$ such that $\mu(\operatorname{supp} x_n) \to 0$ as $n \to \infty$ and $\|Tx_n\| \ge \varepsilon$ for each $n \in \mathbb{N}$. Then $x_n \to 0$ a.e. and by Proposition 1.18, $x_n \xrightarrow{\text{o}} 0$. This contradicts the order-to-norm continuity of T.

For the "if" part, let (x_α) be a net order converging to zero and (u_α) be a net with $|x_\alpha| \le u_\alpha \downarrow 0$. Further, we assume that $\|u_\alpha\| \le 1$. Fix $\varepsilon > 0$ and choose $\delta > 0$ so that for every $x \in L_\infty(\mu)$ with $\|x\| \le 1$ and $\mu(\operatorname{supp} x) < \delta$ we have that $\|Tx\| < \frac{\varepsilon}{2}$. Then by the boundedness of T, there exists $\delta_1 > 0$ so that $\|Tx\| < \frac{\varepsilon}{2}$ whenever $\|x\| < \delta_1$.

For each α, let $B_\alpha = \{\omega \in \Omega : u_\alpha(\omega) > \frac{\delta_1}{2}\}$. Since $\int_\Omega u_\alpha d\mu \ge \frac{\delta_1}{2}\mu(B_\alpha)$ and $\lim_\alpha \int_\Omega u_\alpha d\mu = 0$, we obtain $\lim_\alpha \mu(B_\alpha) = 0$. Thus there exists α_0 such that $\mu(B_\alpha) < \delta$ for every $\alpha \ge \alpha_0$. Let $y_\alpha = x_\alpha - x_\alpha \mathbf{1}_{B_\alpha}$ and $z_\alpha = x_\alpha \mathbf{1}_{B_\alpha}$. The condition $|x_\alpha(\omega)| \le u_\alpha(\omega) \le \frac{\delta_1}{2}$ for each $\omega \in \Omega \setminus B_\alpha$, implies $\|y_\alpha\| \le \frac{\delta_1}{2} < \delta_1$. Hence, $\|Ty_\alpha\| < \frac{\varepsilon}{2}$. On the other hand, since $\operatorname{supp} z_\alpha \subseteq B_\alpha$, we have that $\mu(\operatorname{supp} z_\alpha) < \delta$ for each $\alpha \ge \alpha_0$, and $\|z_\alpha\| \le \|x_\alpha\| \le \|u_\alpha\| \le 1$. Therefore, $\|Tz_\alpha\| < \frac{\varepsilon}{2}$ and so,

$$\|Tx_\alpha\| = \|T(y_\alpha + z_\alpha)\| \le \|T(y_\alpha)\| + \|T(z_\alpha)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for each $\alpha \ge \alpha_0$. Thus, $\lim_\alpha \|Tx_\alpha\| = 0$ and T is order-to-norm continuous. □

The following two corollaries of Lemma 11.52 assert that an order-to-norm continuous operator defined on L_∞ has a separable "essential domain" and hence, a separable range.

Proposition 11.53. *Let $(h_n)_{n=1}^\infty$ be the Haar system on $[0,1]$, X a Banach space and $S, T \in \mathcal{L}(L_\infty, X)$ order-to-norm continuous operators. If $Sh_n = Th_n$ for each $n \in \mathbb{N}$ then $S = T$.*

Proof. Note that for the characteristic function of any dyadic interval $w = \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}$ we have that $Sw = Tw$ (since w belongs to the linear span of the Haar system). Fix any $x \in L_\infty$ and any $\varepsilon > 0$. First, choose a simple function $y = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$, $[0,1] = \bigsqcup_{k=1}^n A_k$ with $\|x - y\| < \varepsilon/(2\|S\| + 2\|T\|)$. Second, using Lemma 11.52, choose a $\delta > 0$ so that for any $u \in L_\infty$, $\|u\| \le 1$ if $\mu(\operatorname{supp} u) < \delta$ then $\|(S - T)u\| < \varepsilon/2$. Third, for each $k = 1, \dots, n$ choose a disjoint union B_k of dyadic intervals so that $B_i \cap B_j = \emptyset$ for $i \ne j$ and $\mu(A_k \triangle B_k) < \delta/n$. Then we obtain

$\mu(\mathrm{supp}(y - z)) < \delta$ for $z = \sum_{k=1}^{n} a_k \mathbf{1}_{B_k}$, and hence, $\|(S - T)(y - z)\| < \varepsilon/2$. Moreover, since z belongs to the span of the Haar system, $(S - T)z = 0$. Hence,

$$\begin{aligned}\|Sx - Tx\| &\le \|(S - T)(x - y)\| + \|(S - T)(y - z)\| \\ &\le \|S - T\|\,\|x - y\| + \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

By arbitrariness of ε, $Sx = Tx$. □

Proposition 11.54. *Let (Ω, Σ, μ) be a finite atomless measure space, X a Banach space and $T \in \mathcal{L}(L_\infty, X)$ an order-to-norm continuous operator. Then the range $T(L_\infty)$ is separable.*

Proof. Using the same arguments as in the proof of Proposition 11.53, one can show that the linear span of the set $\{Th_n : n \in \mathbb{N}\}$ is dense in $T(L_\infty)$. □

Different definitions of a narrow operator on L_∞ for order-to-norm continuous maps

Since $L_\infty(\mu)$ is a vector lattice, we can consider different definitions of narrow operators on $L_\infty(\mu)$: the standard one treating $L_\infty(\mu)$ as a function space, and the vector lattice definition as in Definition 10.1. Let us consider several properties which could mean different types of "narrowness" for an operator $T \in \mathcal{L}(L_\infty(\mu), X)$:

(i) For every $A \in \Sigma$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = \mathbf{1}_A$, $\int_\Omega x \, d\mu = 0$ and $\|Tx\| < \varepsilon$.

(ii) For every $A \in \Sigma$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = \mathbf{1}_A$ and $\|Tx\| < \varepsilon$.

(iii) For every $y \in L_\infty(\mu)^+$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = y$ and $\|Tx\| < \varepsilon$.

(iv) For every $y \in L_\infty(\mu)^+$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = y$, $\int_\Omega x \, d\mu = 0$ and $\|Tx\| < \varepsilon$.

Note that (i) is Definition 1.5, (iii) is the definition of a narrow operator on a vector lattice (Definition 10.1). Properties (ii) and (iv) are their weakest and strongest form, respectively. Thus, either (i) or (iii) implies (ii), and (iv) implies all the other properties.

The equivalence of (iii) and (i) was proved in Proposition 10.2 for operators defined on a Köthe function space with an absolutely continuous norm. Here we prove that all properties (i)–(iv) are equivalent for order-to-norm continuous operators defined on $L_\infty(\mu)$.

Recall that Open problem 1.10 asks whether (ii) implies (i) for every Banach space X and every operator $T \in \mathcal{L}(L_\infty, X)$.

Theorem 11.55. ([72]) *Let X be a Banach space and $T \in \mathcal{L}(L_\infty(\mu), X)$ be an order-to-norm continuous operator. Then all the properties* (i)–(iv) *are equivalent for T.*

Proof. By the above remarks, it is enough to prove the implication (ii) $\Rightarrow$ (iv).

Let $T \neq 0$. Fix any $y \in L_\infty(\mu)^+$ and $\varepsilon > 0$. Without loss of generality we assume that $0 < \|y\| \le 1$. Choose a simple function $u = \sum_{i=1}^m a_i \mathbf{1}_{A_i} \neq 0$ with $a_i \in \mathbb{R}$ and pair-wise disjoint sets $A_i \in \Sigma$ so that $A_i \subseteq \operatorname{supp} y = \operatorname{supp} u$ and

$$\|y - u\| < \frac{\varepsilon}{2\,\|T\|}\,. \tag{11.30}$$

By Lemma 11.52, there exists $\delta > 0$ so that for any $z \in L_\infty(\mu)$, with $\|z\| \le 1$, if $\mu(\operatorname{supp} z) < \delta$ then

$$\|Tz\| < \frac{\varepsilon}{4m\|u\|}\,. \tag{11.31}$$

Consider on Σ the measure μ_y generated by y, i.e. $\mu_y(A) = \int_A y \,\mathrm{d}\mu$ for any $A \in \Sigma$. We shall use the following simple fact whose proof we omit:

> Claim: *There exists $n \in \mathbb{N}$ such that for any $A \in \Sigma$ with $A \subseteq \operatorname{supp} y$, if $\mu_y(A) < \mu(\Omega)/n$ then $\mu(A) < \delta$.*

Choose an n, and for each $i = 1, \dots, m$, divide A_i into $n+1$ parts of equal measure μ_y

$$A_i = \bigsqcup_{k=1}^{n+1} A_{i,k}, \quad \mu_y\big(A_{i,k}\big) = \frac{\mu_y(A_i)}{n+1}\,.$$

For every $i = 1, \dots, m$ and $k = 1, \dots, n$, we use Property (ii) to find $x_{i,k} \in L_\infty(\mu)$ so that $|x_{i,k}| = \mathbf{1}_{A_{i,k}}$ and

$$\big\|Tx_{i,k}\big\| < \frac{\varepsilon}{4nm\|u\|}\,. \tag{11.32}$$

Let $\beta_{i,k} = \int_\Omega y x_{i,k} \,\mathrm{d}\mu$ for $i = 1, \dots, m$ and $k = 1, \dots, n$, and observe that

$$\big|\beta_{i,k}\big| \le \mu_y\big(A_{i,k}\big) = \frac{\mu_y(A_i)}{n+1} \tag{11.33}$$

for each $i = 1, \dots, m$ and $k = 1, \dots, n$.

Fix any i. Using (11.33) and induction on $\ell = 1, \dots, n$, it can be easily shown that there exist signs $\theta_{i,1}, \dots, \theta_{i,\ell} \in \{-1, 1\}$ such that

$$\Big|\sum_{j=1}^{\ell} \theta_{i,j}\beta_{i,j}\Big| \le \frac{\mu_y(A_i)}{n+1}\,.$$

Thus, we choose such signs for $\ell = n$. Then choose $x_{i,n+1} \in L_\infty(\mu)$ satisfying $|x_{i,n+1}| = \mathbf{1}_{A_{i,n+1}}$ and

$$\int_\Omega y x_{i,n+1} \, d\mu = -\sum_{k=1}^{n} \theta_{i,k} \beta_{i,k} \,. \tag{11.34}$$

By the above claim, since

$$\mu_y(A_{i,n+1}) = \frac{\mu_y(A_i)}{n+1} < \frac{\mu(\Omega)}{n} \,,$$

we have that $\mu(A_{i,n+1}) < \delta$ and hence, by (11.31),

$$\|Tx_{i,n+1}\| < \frac{\varepsilon}{4m\|u\|} \,. \tag{11.35}$$

Let $x_i = \sum_{k=1}^n \theta_{i,k} x_{i,k} + x_{i,n+1}$ for $i = 1, \dots, m$. From (11.32) and (11.35) we deduce

$$\|Tx_i\| \le \sum_{k=1}^{n} \|Tx_{i,k}\| + \|Tx_{i,n+1}\| < n \frac{\varepsilon}{4nm\|u\|} + \frac{\varepsilon}{4m\|u\|} = \frac{\varepsilon}{2m\|u\|} \,. \tag{11.36}$$

Moreover, by the above construction, $|x_i| = \mathbf{1}_{A_i}$ and $|\sum_{i=1}^m x_i| = \mathbf{1}_{\operatorname{supp} y}$.

Now we set $x = y \sum_{i=1}^m x_i$ and $v = \sum_{i=1}^m a_i x_i$. Observe that $|x - v| = |y - u|$ a.e. on Ω and hence, $\|x - v\| = \|y - u\|$. Obviously, $|x| = y$. By (11.34),

$$\int_\Omega x \, d\mu = \sum_{i=1}^{m} \int_\Omega y x_i \, d\mu = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \theta_{i,k} \int_\Omega y x_{i,k} \, d\mu + \int_\Omega y x_{i,n+1} \, d\mu \right) = 0 \,.$$

Since $\|x - v\| = \|y - u\|$ and $|a_i| \le \|u\|$, using (11.36) and (11.30) we obtain

$$\|Tx\| \le \|Tv\| + \|T\| \, \|x - v\| \le \sum_{i=1}^{m} |a_i| \, \|Tx_i\| + \frac{\varepsilon}{2} < m\|u\| \frac{\varepsilon}{2m\|u\|} + \frac{\varepsilon}{2} = \varepsilon \,.$$

□

Order-to-norm continuous operators from $L_\infty(\mu)$ to $c_0(\Gamma)$

The main result of this subsection (Theorem 11.56) asserts that every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow, while not every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is AM-compact, where Γ is any infinite set. On the other hand, we construct an example of an order-to-norm continuous operator from L_∞ to c_0 which cannot be extended to a continuous linear operator on any of the spaces L_p with $1 \le p < \infty$. So Theorem 11.56 cannot be deduced from the theorem of V. Kadets and Popov [56] that every operator $T \in \mathcal{L}(L_p, c_0)$ is narrow for any p, $1 \le p < \infty$. Nevertheless, in our proof we follow the ideas of [56].

Theorem 11.56. ([72]) *Every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow, but not every operator of this kind is AM-compact.*

Proof. Fix any $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$, and consider the set

$$K_{\varepsilon,A} = \left\{x \in B_{L_\infty(\mu)} : \|Tx\| \le \varepsilon \quad \text{and} \quad \int_\Omega x \, d\mu = 0\right\}.$$

We claim that $K_{\varepsilon,A}$ is a convex and weakly compact subset of $L_2(\mu)$. The convexity is easy to verify. The only thing that should be explained here is that $K_{\varepsilon,A}$ is weakly closed. By convexity, it is enough to prove that it is norm closed in $L_2(\mu)$. Let $x_n \in K_{\varepsilon,A}$ and $\|x_n - x\|_{L_2(\mu)} \to 0$ as $n \to \infty$. Then, obviously, $\|x\|_{L_\infty(\mu)} \le 1$ and $\int_\Omega x \, d\mu = 0$. By Theorem 11.51, $\|T(x_n - x)\| \to 0$ as $n \to \infty$, which implies that $\|Tx\| \le \varepsilon$.

Thus, by the Krein–Milman theorem, there exists an extreme point $x_0 \in K_{\varepsilon,A}$. We show that $|x_0| = \mathbf{1}_A$. Suppose, to the contrary, that there exists $\delta > 0$ and a subset $B \subseteq A$ with $\mu(B) > 0$ such that $|x_0(\omega)| \le 1 - \delta$ for each $\omega \in B$. Denote by $(e_\gamma)_{\gamma\in\Gamma}$ the unit vector basis for $c_0(\Gamma)$ and by $(e_\gamma^*)_{\gamma\in\Gamma}$ its biorthogonal functionals. Choose a finite set $\Gamma_0 \subset \Gamma$ so that $|e_\gamma^*(Tx_0)| < \varepsilon/2$ for each $\gamma \in \Gamma \setminus \Gamma_0$.

Since $Y_0 = [e_\gamma]_{\gamma\in\Gamma\setminus\Gamma_0}$ has finite codimension in $c_0(\Gamma)$, by Lemma 9.4, the subspace

$$X_0 = T^{-1}Y_0 \cap \left\{x \in L_\infty(\mu) : \int_\Omega x \, d\mu = 0\right\}$$

has finite codimension in $L_\infty(\mu)$. Since $\dim L_\infty(B) = \infty$ we conclude that $L_\infty(B) \cap X_0 \ne \{0\}$, so there exists $y_0 \in L_\infty(B) \cap X_0$ with $y_0 \ne 0$. Choose $\alpha \ne 0$ so that $\|\alpha y_0\|_{L_\infty(\mu)} \le \delta$ and $\|T(\alpha y_0)\| < \varepsilon/2$. Thus, $x_0 \pm \alpha y_0 \in K_{\varepsilon,A}$, which is a contradiction. Thus, by Corollary 11.55, T is narrow.

Finally, we construct an example of an order-to-norm continuous operator $S \in \mathcal{L}(L_\infty, c_0)$ which is not AM-compact. This operator can be used in the obvious manner to obtain an operator of $\mathcal{L}(L_\infty(\mu), c_0(\Gamma))$ with the same properties. Denote by (r_n) the Rademacher system on $[0,1]$ and for every $x \in L_\infty$ set $Sx = (\xi_1, \xi_2, \dots)$ where $\xi_n = \int_\Omega x r_n \, d\mu$ for each $n \in \mathbb{N}$. By Theorem 11.51, since S can be extended to a continuous linear operator $\hat{S} \in \mathcal{L}(L_1, c_0)$, it is order-to-norm continuous. Since the Rademacher system is an order bounded set in L_∞ which is sent by S to a nonrelatively compact subset of c_0, the operator S is not AM-compact. □

Now we show that not every order-to-norm continuous operator from L_∞ to c_0 can be extended to L_p for some $p < \infty$. Therefore, Theorem 11.56 cannot be deduced from the results of [56].

Example 11.57. There exists an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, c_0)$ which cannot be extended to L_p for any $p < \infty$.

Proof. First observe that it is sufficient to construct for each given $p \in [1, +\infty)$, an order-to-norm continuous operator $T = T_p \in \mathcal{L}(L_\infty, c_0)$ that cannot be extended to L_p, because then the desired operator can be easily obtained as the direct sum of such operators for any sequence of p_n tending to infinity.

Therefore, we fix $p \in [1, \infty)$. For any sequence (A_n) of disjoint sets from Σ and any $g \in L_1$, we define for every $x \in L_\infty$

$$Tx = (\xi_1, \xi_2, \ldots), \quad \text{where} \quad \xi_n = \int_{A_n} gx \, \mathrm{d}\mu \, .$$

Since $gx \in L_1$ and $\mu(A_n) \to 0$, we have that $Tx \in c_0$ by the absolute continuity of the Lebesgue integral. Therefore, $T : L_\infty \to c_0$ is a linear operator. Furthermore, given any $x \in L_\infty$, one has that $\int_{A_n} |g||x| \, \mathrm{d}\mu \le \|g\|_{L_1} \|x\|_{L_\infty}$ for every $n \in \mathbb{N}$, hence T is bounded with $\|T\| \le \|g\|_{L_1}$.

To show that T is order-to-norm continuous, we consider any sequence (x_n) in L_∞ order converging to zero (i.e. $|x_n| \le y_n \downarrow 0$ for some sequence (y_n) in L_∞). By Proposition 1.18, (y_n) tends to zero a.e. on $[0, 1]$. Thus, the sequence $(|g| y_n)$ is decreasing and tends to zero a.e. By the Lebesgue theorem, $G_n = \int_\Omega |g| y_n \, \mathrm{d}\mu \to 0$ as $n \to \infty$. On the other hand, for each $n, m \in \mathbb{N}$ we have that

$$\Big| \int_{A_m} g x_n \, \mathrm{d}\mu \Big| \le \int_\Omega |g||x_n| \, \mathrm{d}\mu \le \int_\Omega |g| y_n \, \mathrm{d}\mu = G_n \, ,$$

from which it follows that $\|T x_n\| \le G_n \to 0$ as $n \to \infty$. Thus, by Theorem 11.51, T is order-to-norm continuous.

We now choose a suitable sequence (A_n) and $g \in L_1$ as follows. Let (A_n) be any disjoint sequence in Σ with

$$\mu(A_n) = \frac{1}{\alpha \, n^{3p}}, \quad \text{where} \quad \alpha = \sum_{k=1}^{\infty} \frac{1}{k^{3p}}$$

for each $n \in \mathbb{N}$ and $g = \sum_{n=1}^{\infty} n^{3p-2} \mathbf{1}_{A_n}$. Note that

$$\int_\Omega g \, \mathrm{d}\mu = \sum_{n=1}^{\infty} n^{3p-2} \, \mu(A_n) = \frac{1}{\alpha} \sum_{n=1}^{\infty} n^{3p-2} n^{-3p} = \frac{1}{\alpha} \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6\alpha} < \infty \, .$$

We show that T cannot be extended continuously to L_p in this case. Indeed, putting $x_n = (\mu(A_n))^{-1/p} \mathbf{1}_{A_n}$, we have $\|x_n\|_{L_p} = 1$ and

$$\begin{aligned} \int_{A_n} g x_n \, \mathrm{d}\mu &= n^{3p-2} \big(\mu(A_n)\big)^{-\frac{1}{p}} \mu(A_n) = n^{3p-2} \big(\mu(A_n)\big)^{1-\frac{1}{p}} \\ &= n^{3p-2} \frac{1}{\big(\alpha \, n^{3p}\big)^{1-\frac{1}{p}}} = \alpha^{\frac{1}{p}-1} n \to \infty \ \text{ as } n \to \infty. \end{aligned}$$

□

By Theorem 11.56, every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow.

Let us now discuss possible extensions of this result for operators from L_∞ to ℓ_p.

For $1 \le p < 2$, every operator $T \in \mathcal{L}(L_\infty(\mu), \ell_p)$ is compact. Indeed, every operator $T \in \mathcal{L}(L_\infty(\mu), \ell_p)$ factors through a Hilbert space [77, Corollary 1, p. 285 and Corollary 2, p. 291], and hence is compact by Pitt's theorem [3, Theorem 2.1.4]. Since compact operators are AM-compact, this fact and Theorem 11.50 give that *for $1 \le p < 2$, every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty(\mu), \ell_p)$ is narrow.*

The existence of noncompact operators from L_∞ to ℓ_p with $2 \le p < \infty$ follows immediately from the fact that L_1 contains subspaces isomorphic to ℓ_q for $1 < q \le 2$ [3, Theorem 6.4.18] and so, L_∞ contains quotient spaces isomorphic to ℓ_p for $p \ge 2$.

Moreover, *for $2 \le p < \infty$, there exists an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ which is not AM-compact.* Indeed, the same operator T generated by the Rademacher system as in the last part of the proof of Theorem 11.56 maps L_∞ to ℓ_2, is not AM-compact but is extendable to an operator from L_2 to ℓ_2, so it is order-to-norm continuous. For $p \ge 2$, ℓ_2 is continuously embedded in ℓ_p and so the same example works.

However the following is unknown.

Open problem 11.58. ([72]) Let $2 \le p < \infty$. Is every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ narrow?

A sum of two narrow operators on L_∞ need not be narrow

By Theorem 5.2, if an r.i. space E on $[0, 1]$ has an unconditional basis then every operator $T \in \mathcal{L}(E)$ is a sum of two narrow operators. As noted by Krasikova in [71], for $E = L_\infty$ a weaker assertion is true.

Theorem 11.59. *A sum of two narrow operators on L_∞ need not be narrow.*

The proof uses Theorem 5.2 and a factorization through L_p.

Lemma 11.60. *For each $p \in [1, \infty)$ the identity embedding $J_p : L_\infty \to L_p$ is a sum of two narrow operators $T, S \in \mathcal{L}(L_\infty, L_p)$.*

Proof. Suppose first that $p > 1$. By Theorem 5.2, the identity map $I_p : L_p \to L_p$ is a sum of two narrow operators $P, Q \in \mathcal{L}(L_p)$. Let $T = P \circ J_p$ and $S = Q \circ J_p$. We claim that T and S are narrow operators. Indeed, given any measurable set $A \subseteq [0, 1]$ and $\varepsilon > 0$, let $x \in L_p$ so that $x^2 = \mathbf{1}_A$, $\int_{[0,1]} x \, d\mu = 0$ and $\|Px\| < \varepsilon$. Since $x \in L_\infty$, we have that $\|Tx\| = \|Px\| < \varepsilon$, thus, T is narrow. Likewise, S is narrow. Furthermore,

$$T + S = P \circ J_p + Q \circ J_p = (P + Q) \circ J_p = J_p \, .$$

Now let $p = 1$. Observe that $J_1 = J \circ J_2$ where $J : L_2 \to L_1$ is the identity embedding. Let $J_2 = T + S$, where $T, S : L_\infty \to L_2$ are narrow. By Proposition 1.8, $J \circ T$ and $J \circ S$ are narrow operators from L_∞ to L_1. Thus,

$$J_1 = J \circ (T + S) = J \circ T + J \circ S$$

is the desired representation. □

Proof of Theorem 11.59. Let $U : L_2 \to L_\infty$ be any isomorphic embedding and $T, S \in \mathcal{L}(L_\infty, L_2)$ be any narrow operators such that $T + S = J_2$ where $J_2 : L_\infty \to L_2$ is the identity embedding (see Lemma 11.60). By Proposition 1.8, the operators $U \circ T$ and $U \circ S$ are narrow members of $\mathcal{L}(L_\infty)$. We show that their sum V is not narrow. Indeed,

$$V = U \circ T + U \circ S = U \circ J_2 ,$$

and hence for each $x \in L_\infty$ with $x^2 = \mathbf{1}_{[0,1]}$ we have

$$\|Vx\| \geq \|U^{-1}\|^{-1} \|J_2 x\| = \|U^{-1}\|^{-1} \int_{[0,1]} x^2 \, \mathrm{d}\mu = \|U^{-1}\|^{-1} ,$$

and thus V is not narrow. □

Open problem 11.61. Is the identity operator on L_∞ a sum of two narrow operators?

If yes, then by Proposition 1.8, every operator from $\mathcal{L}(L_\infty)$ equals a sum of two narrow operators.

By Theorem 10.5, the set of all narrow regular operators on L_∞ is not a band in the vector lattice of all regular linear operators on L_∞. However, we do not know whether it is a linear subspace.

Open problem 11.62.[1] Is a sum of two regular narrow operators from $\mathcal{L}(L_\infty)$ narrow?

Even the following is unknown (see also Open problem 5.6).

Open problem 11.63. Is a sum of two narrow functionals from L_∞^* narrow?

Recall that we do not know whether the set of all regular order narrow operators on L_∞ is a band (see Open problem 10.43).

[1] This problem has been recently solved in the negative by Mykhaylyuk and the first named author in the paper "On sums of narrow operators on Köthe function spaces". Preprint

11.5 Narrow 2-homogeneous polynomials

In this short section we observe that every 2-homogeneous scalar polynomial on L_p, $1 \le p < 2$ is narrow. Polynomials on Banach spaces, according to Aron [10], play a crucial "intermediate" role between linear mappings and arbitrary continuous or differentiable functions. They have been actively studied by many mathematicians. For a brief engaging introduction to the subject we refer the reader to [10], and for the general work and bibliography on polynomials in infinite dimensional Banach spaces to [30]. Here we present just one result concerning narrowness of polynomials. We are not aware of any other results in this direction, but we feel that there may be some attractive problems in this area.

Let X, Y be Banach spaces and $n \in \mathbb{N}$. A map $L : X^n \to Y$ is called *polylinear* (more exactly, *n-linear*) if for any k, $1 \le k \le n$, and all $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in X$ the map $T : X \to Y$ defined by

$$Tx = L(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)\,,$$

is a linear continuous map. If $L : X^n \to Y$ is a polylinear map then the diagonal map $Px = L(x, x, \dots, x)$ is called an *n-homogeneous polynomial*. An n-homogeneous polynomial $P : X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is called an *n-homogeneous scalar polynomial*.

In particular, for $n = 1$, the 1-homogeneous scalar polynomials are exactly the linear functionals. Observe that for any $T \in \mathcal{L}(X, X^*)$ the map

$$P(x) = (Tx, x) = Tx(x), \quad x \in X \tag{11.37}$$

is a 2-homogeneous scalar polynomial. By [40], the converse is also true: for every 2-homogeneous scalar polynomial P on X, there exists $T \in \mathcal{L}(X, X^*)$ that generates P by (11.37).

Proposition 11.64. *Let $1 \le p < 2$. Then every 2-homogeneous scalar polynomial $P : L_p \to \mathbb{K}$ is narrow.*

Proof. Assume first that $1 < p < 2$ and set $q = p/(p-1)$. Let $T \in \mathcal{L}(L_p, L_q)$. By Theorem 9.7, T is narrow. This yields that the polynomial P is also narrow, defined by (11.37), because of the following inequalities:

$$\begin{aligned}|P(x)| &= \Big| \int_{[0,1]} Tx \cdot x \, \mathrm{d}\mu \Big| \le \Big(\int_{[0,1]} |Tx|^q \, \mathrm{d}\mu \Big)^{1/q} \Big(\int_{[0,1]} |x|^p \, \mathrm{d}\mu \Big)^{1/p} \\ &= \|Tx\| \cdot \|x\|\,.\end{aligned}$$

Assume now that $p = 1$. Let $P : L_1 \to \mathbb{K}$ be any 2-homogeneous scalar polynomial. Fix any $r \in (1, 2)$ and consider a 2-homogeneous scalar polynomial $P_1 : L_r \to \mathbb{K}$ defined by

$$P_1(x) = P(x), \quad x \in L_r$$

where P_1 is well defined by the continuity of the inclusion $L_r \subseteq L_1$. By part one of the proof, P_1 is narrow, and hence, so is P. □

The elegant proof shown above of Proposition 11.64 for $p = 1$ is due to V. Kadets.

Observe that the assertion of Proposition 11.64 is false for $p \geq 2$ because of the following obvious example

$$P(f) = (f, f) = \int_{[0,1]} f^2 \, \mathrm{d}\mu .$$

Chapter 12

Open problems

In this chapter, for the convenience of the reader, we list all the open problems that were stated throughout the book. In this chapter we number them consecutively and, for easy reference, in parenthesis we provide the original number from the text. We refer the reader to the statement in the text for comments on the context of the listed problems and for any relevant partial results. In square brackets we reference the original source where the problems were first posed. If that reference is not present, it means that this book is the first to pose the problem.

We group the problems by subject.

Definition of a narrow operator

Open problem 1. (1.10) [72] Does Definition 1.5 remain the same for $E = L_\infty$ if the condition on a sign to be of mean zero is omitted?

Open problem 2. (10.3) [117] Are Definitions 1.5 and 10.1 equivalent for every Köthe–Banach space E on a finite atomless measure space, and every Banach space X? What if $E = L_\infty$?

Strict singularity versus narrowness

Open problem 3. (2.6 and 7.1(a)) [110] Let E be a Köthe–Banach space with an absolutely continuous norm, not isomorphic to an $L_1(\mu)$-space, and X be a Banach space. Is every strictly singular operator $T \in \mathcal{L}(E, X)$ narrow?

Open problem 4. (2.7 and 7.1(b)) [110] Let E be a Köthe–Banach space with an absolutely continuous norm, and X be a Banach space. Is every ℓ_2-strictly singular operator $T \in \mathcal{L}(E, X)$ narrow?

Open problem 5. (7.1(b)) Let T be an operator from E to X. Does T have to be narrow, provided that T is Z-strictly singular for an appropriately chosen infinite dimensional subspace Z of E?

Open problem 6. (7.1(c)) Let T be an operator from E to X. Does T have to be narrow, provided that T is non-Enflo, that is, T is E-strictly singular?

Strictly narrow operators

Open problem 7. (2.17) Let E be a Köthe F-space with an absolutely continuous norm on (Ω, Σ, μ) and X be an F-space. Suppose that $\operatorname{dens} E(A) > \operatorname{dens} X$ for every $A \in \Sigma^+$. Does it follow that every operator $T \in \mathcal{L}(E, X)$ is strictly narrow?

Invariant subspaces for narrow operators

Open problem 8. (2.18) Let E be a Köthe–Banach space on (Ω, Σ, μ). Does every narrow operator $T \in \mathcal{L}(E)$ have a nontrivial invariant subspace?

Numerical index of a Banach space

Open problem 9. (5.12) [90] Let $1 < p < \infty$, $p \neq 2$.

$$n_{\mathrm{nar}}(L_p) = \inf\{v(T) \,:\, T \in \mathcal{L}(L_p),\ \|T\| = 1,\ T \text{ is narrow}\}.$$

Does $n(L_p) = n_{\mathrm{nar}}(L_p)$?

Generalizations of results from the setting of L_1-spaces to L_p-spaces

Open problem 10. (7.51) Characterize Banach spaces X for which a sum of two narrow operators in $\mathcal{L}(L_1, X)$ narrow.

Open problem 11. (7.52) [102] Suppose an operator $T \in \mathcal{L}(L_p)$, $1 < p \leq 2$, is such that for every $A \in \Sigma^+$ the restriction $T|_{L_p(A)}$ is not an isomorphic embedding. Does it follow that T is narrow?

Open problem 12. (7.81) Suppose $1 < p < \infty$, $p \neq 2$. Let X be a Banach space and suppose that $T \in \mathcal{L}(L_p, X)$ fixes a copy of L_p. Do there exist $A_0 \in \Sigma^+$ and an atomless sub-σ-algebra Σ_0 of $\Sigma(A_0)$ such that the restriction $T|_{L_p(A_0,\Sigma_0)}$ is an into isomorphism? What if $X = L_p$?

Weak embeddings of L_1

Open problem 13. (8.7) [101]

(a) Suppose that L_1 sign-embeds in X. Does L_1 G_δ-embed in X?

(b) Suppose that L_1 G_δ-embeds in X. Does L_1 sign-embed in X?

(c) Suppose that L_1 sign-embeds in X. Does L_1 semi-embed in X?

Open problem 14. (8.10) [101]

(a) Assume that L_1 semi-embeds in X, and Y is a subspace of X. Does L_1 semi-embed either in Y or in X/Y?

(b) Assume that L_1 G_δ-embeds in X, and Y is a subspace of X. Does L_1 G_δ-embed either in Y or in X/Y?

Regularly complemented subspaces of L_p

Open problem 15. (10.45) [93] Let $1 \le p < \infty$, $p \ne 2$. Is every regularly complemented subspace of L_p isomorphic to either ℓ_p or L_p?

Narrow operators on vector lattices

Open problem 16. (10.42) [93] Is Theorem 10.40 true for regular operators, which are not order continuous?

Open problem 17. (10.43) [93] Is the set of all order narrow regular operators $T : L_\infty \to L_\infty$ a band in the vector lattice $L_r(L_\infty)$ of all regular linear operators on L_∞?

Open problem 18. (10.46) [93] Let E be an order continuous Banach lattice and $T \in L_r(E)$ be a regular operator. Suppose that for each band $F \subseteq E$ the restriction $T|_F$ is not an isomorphic embedding. Must T be narrow?

Rich subspaces

Open problem 19. (6.14) (Semenov, [118]) Let E be an r.i. space on a finite atomless measure space, $E \ne L_2$. Does there exist a constant $k_E > 1$ such that if $P \ne I$ is a projection onto a rich subspace of E then $\|P\| \ge k_E$?

Open problem 20. (7.54) [110, p. 73] Let E be an r.i. Banach space on $[0,1]$, $E \ne L_1$. Let X be a subspace of E such that $\rho(L_1(A), X) = 0$ for every $A \in \Sigma^+$. Must X be rich?

Hereditarily narrow operators

Open problem 21. (11.9) Let $2 < p < \infty$. Is the orthogonal projection P, defined by (11.1), from L_p onto the span R of the Rademacher system (r_n), hereditarily narrow?

Open problem 22. (11.6) Let $1 < p < 2$. Is every hereditarily narrow operator $T \in \mathcal{L}(L_p)$ non-Enflo?

Narrow operators on $L_\infty(\mu)$

Open problem 23. (11.47) [117] Does there exist a narrow functional $f \in L_\infty^*$ which is not strictly narrow?

Open problem 24. (11.58) [72] Let $2 \le p < \infty$. Is every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ narrow?

Open problem 25. (11.61) Is the identity operator on L_∞ a sum of two narrow operators?

Open problem 26. (5.6) Is the sum of two narrow operators from $\mathcal{L}(L_\infty)$, at least one of which is compact, narrow?

Open problem 27. (11.63) [117] Is the sum of two narrow functionals from L_∞^* narrow?

C-Narrow operators on $C[0,1]$

Open problem 28. (11.45) Is the subspace of $\mathcal{L}(C[0,1])$ consisting of all C-narrow operators, complemented in $\mathcal{L}(C[0,1])$?

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